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Ph.D. Thesis

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Sufficient Conditions for a Maltsev Product of Two Varieties To Be a Variety

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Abstract

The Maltsev product $\mathcal{V} \circ \mathcal{W}$ of varieties $\mathcal{V}$ and $\mathcal{W}$ of the same type, is the class of all algebras A that have a congruence θ such that the quotient A/θ belongs to $\mathcal{W}$ and every congruence class of θ which is a subalgebra of A belongs to $\mathcal{V}$. The class $\mathcal{V} \circ \mathcal{W}$ may not be a variety. We identify a class of varieties that behave well as the second factor of the Maltsev product. We call them *term idempotent varieties*. They include in particular all idempotent varieties. The main result of this work is a sufficient condition for the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ to be a variety. We use this sufficient condition to derive a number of other sufficient conditions. One of the most interesting of these results states that the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of any congruence permutable variety $\mathcal{V}$ and any term idempotent variety $\mathcal{W}$ is a variety. We provide an equational base for the variety generated by a Maltsev product of two varieties.

Keywords

Maltsev product, variety, term idempotent variety, equational base.

Streszczenie

Produkt Malceva $\mathcal{V} \circ \mathcal{W}$ rozmaitości $\mathcal{V}$ i $\mathcal{W}$ tego samego typu to klasa złożona ze wszystkich algebr A, które posiadają kongruencję θ, taką że iloraz A/θ należy do $\mathcal{W}$, a każda klasa abstrakcji, która jest podalgebrą A, należy do $\mathcal{V}$. Klasa $\mathcal{V} \circ \mathcal{W}$ może nie być rozmaitością. Zidentyfikowaliśmy klasę rozmaitości, które zachowują się dobrze jako drugi czynnik produktu Malceva. Nazwaliśmy je *rozmaitościami termowo idempotentnymi*. Do tej klasy należą w szczególności wszystkie rozmaitości idempotentne. Głównym wynikiem tej pracy jest warunek dostateczny na to, aby produkt Malceva $\mathcal{V} \circ \mathcal{W}$ rozmaitości $\mathcal{V}$ oraz rozmaitości termowo idempotentnej $\mathcal{W}$ był rozmaitością. Z tego warunku dostatecznego wyprowadziliśmy serię pochodnych warunków dostatecznych. Jeden z najciekawszych mówi, że dla dowolnej rozmaitości $\mathcal{V}$, która ma przemienne kongruencje, oraz dowolnej rozmaitości termowo idempotentnej $\mathcal{W}$, produkt Malceva $\mathcal{V} \circ \mathcal{W}$ jest rozmaitością. Podaliśmy również bazę równościową dla rozmaitości generowanej przez produkt Malceva dwóch rozmaitości.

Słowa kluczowe

produkt Malceva, rozmaitość, rozmaitość termowo idempotentna, baza równościowa.

Contents

1 Introduction

For a class $\mathcal{K}$ of algebras, the *Maltsev $\mathcal{K}$–product* $\mathcal{C} \circ_{\mathcal{K}} \mathcal{D}$ of classes $\mathcal{C}, \mathcal{D} \subseteq \mathcal{K}$, is the class that consists of all algebras $A \in \mathcal{K}$ which have a congruence θ, such that the quotient A/θ belongs to $\mathcal{D}$ and each congruence class of θ which is a subalgebra of A that belongs to $\mathcal{K}$, belongs to $\mathcal{C}$, i.e.

$$\mathcal{C} \circ_{\mathcal{K}} \mathcal{D} = \{A \in \mathcal{K} \mid \exists\theta \ A/\theta \in \mathcal{D},\ \forall a \in A \ (a/\theta \leq A,\ a/\theta \in \mathcal{K} \ \Rightarrow \ a/\theta \in \mathcal{C})\}. \tag{1.1}$$

Let $\mathcal{A}$ be the class of all algebras of a given type. The Maltsev $\mathcal{A}$–product $\mathcal{C} \circ_{\mathcal{A}} \mathcal{D}$ will be called the *absolute Maltsev product* or simply the *Maltsev product* and will be denoted by $\mathcal{C} \circ \mathcal{D}$. A *variety* is a class of algebras which is closed under subalgebras, arbitrary direct products, and homomorphic images. Maltsev $\mathcal{K}$–products were introduced by Maltsev [15] in order to extend the product of varieties of groups defined by Neumann [17] to arbitrary classes of algebras. If $\mathcal{G}$ is the variety of groups, then for any subvarieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{G}$, the Maltsev $\mathcal{G}$–product $\mathcal{V} \circ_{\mathcal{G}} \mathcal{W}$ coincides with the Neumann product of $\mathcal{V}$ and $\mathcal{W}$.

Maltsev $\mathcal{K}$–products offer a context for different algebraic constructions such as group extensions, semilattices of semigroups (see [12]), or semilattice sums of algebras (see [20]). For any classes $\mathcal{C}$ and $\mathcal{D}$ of groups, the Maltsev $\mathcal{G}$–product $\mathcal{C} \circ_{\mathcal{G}} \mathcal{D}$ consists of all extensions of groups in $\mathcal{C}$ by groups in $\mathcal{D}$. If $\mathcal{S}g$ is the variety of semigroups and $\mathcal{S}$ is the variety of semilattices, then for any class $\mathcal{C}$ of semigroups, the Maltsev $\mathcal{S}g$–product $\mathcal{C} \circ_{\mathcal{S}g} \mathcal{S}$ consists of all semilattices of $\mathcal{C}$–semigroups. If a type Ω obeys certain mild conditions, then there exists a unique variety $\mathcal{S}_\Omega$ of the type Ω which is equivalent to the variety of semilattices. For any class $\mathcal{C}$ of algebras of such a type, the Maltsev product $\mathcal{C} \circ \mathcal{S}_\Omega$ consists of all semilattice sums of algebras in $\mathcal{C}$.

Maltsev $\mathcal{K}$–products are applied to describe the structure of classes of algebras. E.g. let $\mathcal{B}$ be the variety of bands (idempotent semigroups) and $\mathcal{R}b$ be the variety of rectangular bands (bands that satisfy the identity $(x \cdot y) \cdot z = x \cdot z$). By a theorem of Clifford and McLean [12, Thm. 3.1], every band is a semilattice of rectangular bands, so $\mathcal{B}$ can be decomposed as the Maltsev $\mathcal{B}$–product $\mathcal{R}b \circ_{\mathcal{B}} \mathcal{S}$. Neumann [17] proved that every variety of groups can be uniquely decomposed as a Maltsev $\mathcal{G}$–product of indecomposable varieties. Analogous or similar results have been obtained e.g. for varieties of Brouwerian semilattices by Köhler [14], varieties of generalized interior algebras by Blok and Köhler [4], and varieties of lattices by Grätzer and Kelly [9].

A *prevariety* is a class of algebras which is closed under subalgebras, arbitrary direct products, and isomorphic images (see [21, Def. 1.5.11]). A *quasivariety* is a prevariety which is additionally closed under directed colimits (see [21, p. 158]). Every variety is a quasivariety. In the basic paper [15], Maltsev investigated sufficient conditions for Maltsev $\mathcal{K}$–products of prevarieties, quasivarieties, and varieties to be prevarieties, quasivarieties, and varieties respectively.

Theorem 1.1. [15, Cor. 5] *Let $\mathcal{P}$ be a prevariety. Every Maltsev $\mathcal{P}$–product of prevarieties is a prevariety.*

Theorem 1.2. [15, Cor. 5] *Let $\mathcal{Q}$ be a quasivariety. If the type of $\mathcal{Q}$ is finite, then every Maltsev $\mathcal{Q}$–product of quasivarieties is a quasivariety.*

A variety $\mathcal{V}$ is *congruence permutable* if for any algebra $A \in \mathcal{V}$ and any pair of congruences θ and ψ of A, one has $\theta \circ \psi = \psi \circ \theta$. A variety $\mathcal{V}$ is *polarized* if every nonempty algebra of $\mathcal{V}$ contains exactly one idempotent element.

Theorem 1.3. [15, Thm. 7] *Let $\mathcal{U}$ be a variety. If $\mathcal{U}$ is congruence permutable and polarized, then every Maltsev $\mathcal{U}$–product of varieties is a variety.*

Iskander [13] gave a sufficient condition for a Maltsev $\mathcal{U}$–product of varieties to be a variety, which is weaker than the condition presented in Theorem 1.3. A variety $\mathcal{U}$ is *weakly congruence permutable* if every nonempty algebra of $\mathcal{U}$ contains an idempotent element and there exist terms $p(x, y, z)$ and $t(x)$, such that $\mathcal{U}$ satisfies the identities $p(t(x), y, y) = t(x)$ and $p(t(x), t(x), y) = y$.

Theorem 1.4. [13, Thm. 4.11] *Let $\mathcal{U}$ be a variety. If $\mathcal{U}$ is weakly congruence permutable, then every Maltsev $\mathcal{U}$–product of varieties is a variety.*

For any weakly congruence permutable variety $\mathcal{U}$ and any subvarieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$, Iskander gave an equational base relative to $\mathcal{U}$ for the variety $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$.

Some authors investigated sufficient conditions for a Maltsev $\mathcal{U}$–product of varieties to be a variety for specific varieties $\mathcal{U}$. Grätzer and Kelly [10] considered Maltsev $\mathcal{L}$–products for the variety $\mathcal{L}$ of all lattices. They showed that if a variety of lattices $\mathcal{V}$ is closed under a certain construction called *gluing*, then the Maltsev $\mathcal{L}$–product $\mathcal{V} \circ_{\mathcal{L}} \mathcal{D}$ is a variety, where $\mathcal{D}$ is the variety of distributive lattices.

The sufficient condition of Iskander cannot be applied to the case of the absolute Maltsev product of varieties, because the variety of all algebras of a given type is not weakly congruence permutable. Bergman [1] gave a sufficient condition that applies to this case.

Theorem 1.5. [1, Cor. 2.3] *Let $\mathcal{V}$ and $\mathcal{W}$ be idempotent varieties. If the join $\mathcal{V} \vee \mathcal{W}$ is congruence permutable, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

The focus of this work are sufficient conditions for the absolute Maltsev product $\mathcal{V} \circ \mathcal{W}$ of varieties $\mathcal{V}$ and $\mathcal{W}$ to be a variety. We obtain a Maltsev style sufficient condition which requires the existence of certain terms such that $\mathcal{V}$ and $\mathcal{W}$ satisfy certain identities involving these terms. We then apply this general sufficient condition to derive a number of other sufficient conditions. One of these sufficient conditions extends Theorem 1.5 to the case when $\mathcal{V} \vee \mathcal{W}$ is a congruence 3–permutable variety (i.e. for every pair of congruences θ and ψ of an algebra $A \in \mathcal{V} \vee \mathcal{W}$, one has $\theta \circ \psi \circ \theta = \psi \circ \theta \circ \psi$). We provide an equational base for $\mathcal{V} \circ \mathcal{W}$ in case when it is a variety. More generally, for varieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$, we provide an equational base relative to $\mathcal{U}$ for the variety generated by $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$. In order to achieve the results of this work, we define the notions of a *term idempotent*, a *term idempotent identity*, and a *term idempotent variety*. We develop a theory of these objects.

Chapter 2 describes the preliminary notions and theorems of universal algebra which we use throughout this work. In Chapter 3 we define a special kind of terms for a given variety $\mathcal{V}$ which we call *term idempotents* of $\mathcal{V}$. These are the terms t such that in any algebra $A \in \mathcal{V}$, all values of the corresponding term operation t^A are idempotent elements of A. The name is motivated by the fact that a term t is a term idempotent of a variety $\mathcal{V}$ if and only if (the equivalence class of) t is an idempotent element of the free algebra of $\mathcal{V}$ over countably infinitely many generators. An example of a term idempotent is the term $x \cdot x^{-1}$ in varieties of groups. Unary term idempotents were already introduced by Iskander [13] under the name of *unit terms*. However the results of this work demonstrate the utility of considering term idempotents of arbitrary arities. In the study of Maltsev products an important role is played by congruence classes which are subalgebras, as the definition (1.1) suggests. Term idempotents can be used to keep track of such congruence classes. We describe the properties of term idempotents.

In Chapter 4, we apply term idempotents to construct for any varieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$, a set of identities that defines relative to $\mathcal{U}$ the variety generated by the prevariety $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$. In particular this yields an equational base for the variety generated by the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of any varieties $\mathcal{V}$ and $\mathcal{W}$ of the same type. Both sides of every identity in this equational base are term idempotents of $\mathcal{W}$. This motivates the following definition. If an identity $u = v$ is true in a variety $\mathcal{V}$ and both u and v are term idempotents of $\mathcal{V}$, then we will call $u = v$ a *term idempotent identity* of $\mathcal{V}$. Identities of this kind are examined in Chapter 5. In Chapter 6 we use term idempotent identities to define *term idempotent varieties*. These are varieties $\mathcal{V}$ such that every nontrivial identity true in $\mathcal{V}$ is a term idempotent identity of $\mathcal{V}$. We present a number of examples of term idempotent varieties. In particular every variety which is idempotent is also term idempotent. We investigate the properties of individual term idempotent varieties and of the class of all such varieties of a given type.

Chapter 7 is concerned with replica congruences. For a variety $\mathcal{V}$ and an algebra A of the same type, the $\mathcal{V}$–replica congruence of A is the smallest congruence of A such that the corresponding quotient belongs to $\mathcal{V}$. The importance of replica congruences in the theory of Maltsev products of varieties comes from the fact that in order to check whether an algebra belongs to the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of varieties $\mathcal{V}$ and $\mathcal{W}$, the only congruence θ that one needs to consider in the definition (1.1) is the $\mathcal{W}$–replica congruence. We provide an explicit construction of the replica congruence. We exploit this construction to obtain several results about $\mathcal{W}$–replica congruences of algebras in the variety generated by $\mathcal{V} \circ \mathcal{W}$. We also apply it to prove a characterization of term idempotent varieties as varieties $\mathcal{W}$ such that for every algebra A, all congruence classes of the $\mathcal{W}$–replica congruence of A which are not subalgebras of A are singletons. Compare this to idempotent varieties $\mathcal{W}$ for which all congruence classes of any $\mathcal{W}$–replica congruence are subalgebras. This property of term idempotent varieties ensures that they behave almost as well as idempotent varieties in the role of the second factor of the Maltsev product.

In Chapter 8, we prove the main result of this work – a sufficient condition for the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ to be a variety. This result is the culmination of research partly published in [3], [18], and [19]. In this work we employ a new proof strategy that allows us to obtain a sufficient condition which is considerably weaker than the one presented in [19]. In Chapter 9, we obtain other sufficient conditions as corollaries of the main theorem and we provide examples of their application. One of the most interesting of these results states that the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of any congruence permutable variety $\mathcal{V}$ and any term idempotent variety $\mathcal{W}$ is a variety.

2 Preliminaries

The books [2], [5], [16], and [21] will be used as references for basic notions of universal algebra. A *similarity type*, or briefly a *type*, is a set Ω of symbols of basic operations together with an assignment of a natural number called *arity* to each symbol $f \in \Omega$. When we consider different algebras, varieties, or terms in the same context, then unless stated otherwise, we implicitly assume that they are of the same type. In particular we only consider Maltsev products of varieties of the same type. We denote an algebra and its universe by the same symbol.

We denote by $T_\Omega(X)$ the set of terms of a type Ω over a set X of variables. We write $T(X)$ if there is no risk of confusion. We denote finite (indexed) sets $\{x_1, \ldots, x_n\}$ of (pairwise distinct) variables by $\boldsymbol{x}$. We denote the set of variables that occur in a term t by var(t). If var(t) has n elements, then we say that t is *n–ary*. If for a term t we write $t(\boldsymbol{x})$, it means that var$(t) \subseteq \boldsymbol{x}$. We will use the first infinite ordinal ω as the standard countably infinite set of variables. Unless stated otherwise, all sets $\boldsymbol{x}$ of variables that we consider are subsets of ω and all terms that we consider belong to $T(\omega)$.

We denote finite (indexed) sets $\{a_1, \ldots, a_n\}$ of elements of a given algebra by $\boldsymbol{a}$. We denote the set of substitutions $\boldsymbol{a}$ of elements of an algebra A for variables of $\boldsymbol{x}$ by $A^{\boldsymbol{x}}$. A term t and a finite set $\boldsymbol{x} \supseteq$ var(t) of variables determine the term operation $t^A(\boldsymbol{x})\colon A^{\boldsymbol{x}} \to A$ of an algebra A. We denote the value of a term operation $t^A(\boldsymbol{x})$ on elements $\boldsymbol{a} \in A^{\boldsymbol{x}}$ by $t(\boldsymbol{a})$.

An element a of an algebra A is an *idempotent element* or an *idempotent* of A if for every $f \in \Omega$, one has $f(a, \ldots, a) = a$. Equivalently, a is an idempotent of A if $\{a\}$ is a subalgebra of A. An algebra A is *idempotent* if all elements of A are idempotents. Note that for a type Ω that contains symbols of constants (nullary basic operations), if an algebra A of the type Ω has an idempotent, then this idempotent is unique and it coincides with every constant of A.

2.1 Varieties and identities

We will say that a class $\mathcal{C}$ of algebras is of a type Ω if all algebras in $\mathcal{C}$ are of the type Ω. We will assume that all classes of algebras which we consider in this work consist of algebras of the same type. A *prevariety* is a class of algebras which is closed under subalgebras, arbitrary direct products, and isomorphic images (see [21, Def. 1.5.11]). A *variety* is a prevariety which is closed under homomorphic images (see [5, Def. 9.3]). Algebras that belong to a variety $\mathcal{V}$ are called *$\mathcal{V}$–algebras*. A variety $\mathcal{V}$ is *idempotent* if every $\mathcal{V}$–algebra is idempotent. The smallest variety that contains a class $\mathcal{C}$ of algebras is called the variety *generated* by $\mathcal{C}$. It follows from Tarski's theorem (see [5, Thm. 9.5]) that the variety generated by a prevariety $\mathcal{P}$ coincides with the class $\mathsf{H}(\mathcal{P})$ of all homomorphic images of algebras in $\mathcal{P}$.

Varieties of a given type Ω form a complete lattice with respect to the class inclusion. For varieties $\mathcal{V}_i$, $i \in I$, the meet $\bigwedge_{i\in I} \mathcal{V}_i$ is the intersection $\bigcap_{i\in I} \mathcal{V}_i$ and the join $\bigvee_{i\in I} \mathcal{V}_i$ is the variety generated by the union $\bigcup_{i\in I} \mathcal{V}_i$. The maximum variety is the variety $\mathcal{A}_\Omega$ of all algebras of the type Ω. The minimum variety is the *trivial* variety $\mathcal{T}_\Omega$ that consists of all single-element algebras of the type Ω and of the empty algebra in case when Ω has no symbols of constants. We will write $\mathcal{A}$ and $\mathcal{T}$ instead of $\mathcal{A}_\Omega$ and $\mathcal{T}_\Omega$ if there is no risk of confusion. Lattices of varieties are discussed in [2, Sec. 4.5].

An *identity* or an *equation* is a formula of the form $u = v$, where u and v are terms. The terms u and v are called the *left-hand side* and the *right-hand side* of the identity respectively. An algebra A *satisfies* an identity $u(\boldsymbol{x}) = v(\boldsymbol{x})$ if $u(\boldsymbol{a}) = v(\boldsymbol{a})$ for all $\boldsymbol{a} \in A^{\boldsymbol{x}}$. If all $\mathcal{V}$–algebras satisfy an identity σ, then we say that σ is *true* in $\mathcal{V}$ or that $\mathcal{V}$ *satisfies* σ and we write $\mathcal{V} \models \sigma$. For a set Σ of identities, we write $\mathcal{V} \models \Sigma$ if $\mathcal{V} \models \sigma$ for all $\sigma \in \Sigma$. An identity is *trivial* if it is of the form $t = t$ for some term t; otherwise it is *nontrivial*. Every variety satisfies all trivial identities. If $\mathcal{V} \models u = v$, then we say that terms u and v are *equivalent* in $\mathcal{V}$ or *$\mathcal{V}$-equivalent*.

A class $\mathcal{C}$ of algebras is *defined* by a set Σ of identities if for every algebra A,

$$A \in \mathcal{C} \quad\Longleftrightarrow\quad A \models \Sigma.$$

Theorem 2.1. [5, Thm. 11.9] *A class of algebras is a variety iff it is defined by a set of identities.*

A set of identities that defines a variety $\mathcal{V}$ is called an *equational base* for $\mathcal{V}$. A subvariety

$\mathcal{W}$ of a variety $\mathcal{V}$ is *defined relative to* $\mathcal{V}$ by a set Σ of identities if for every $\mathcal{V}$–algebra A,

$$A \in \mathcal{W} \quad \Longleftrightarrow \quad A \models \Sigma.$$

In this case for any equational base $\Sigma_{\mathcal{V}}$ of $\mathcal{V}$, the union $\Sigma \cup \Sigma_{\mathcal{V}}$ is an equational base for $\mathcal{W}$.

A *magma* is an algebra with a single basic operation that is binary. Below, we introduce several varieties of magmas which we will frequently encounter throughout the text (see [12]):

(1) $\mathcal{S}g$ of all *semigroups* defined by the *associative law* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(2) $\mathcal{B}$ of all *bands* defined relative to $\mathcal{S}g$ by the *idempotent law* $x \cdot x = x$,

(3) $\mathcal{S}$ of all *semilattices* defined relative to $\mathcal{B}$ by the *commutative law* $x \cdot y = y \cdot x$,

(4) $\mathcal{R}b$ of all *rectangular* bands defined relative to $\mathcal{B}$ by the identity $(x \cdot y) \cdot z = x \cdot z$,

(5) $\mathcal{L}z$ of all *left-zero* semigroups defined relative to $\mathcal{S}g$ (or to $\mathcal{B}$) by the identity $x \cdot y = x$,

(6) $\mathcal{R}z$ of all *right-zero* semigroups defined relative to $\mathcal{S}g$ (or to $\mathcal{B}$) by the identity $x \cdot y = y$.

An identity σ is a *consequence* of a set Σ of identities if whenever an algebra satisfies the identities in Σ, it also satisfies σ. One may infer the consequences of a given set of identities using the following rules of inference of *equational logic* (see [5, Sec. 14]).

(e1) Infer $p = p$, for any term p.

(e2) From $u = v$ infer $v = u$.

(e3) From $u = v$ and $v = w$ infer $u = w$.

(e4) From $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ infer $u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n)$,
for any terms $t_1, \ldots, t_n$.

(e5) From $u = v$ infer $t(u, \boldsymbol{x}) = t(v, \boldsymbol{x})$, for any term $t(y, \boldsymbol{x})$.

Theorem 2.2. [5, Thm. 14.19] *An identity σ is a consequence of a set Σ of identities iff σ can be inferred from the identities in Σ using the rules of inference of equational logic.*

The set $\mathrm{Id}(\mathcal{V})$ of all identities true in a variety $\mathcal{V}$ is called the *equational theory of* $\mathcal{V}$. A set of identities is called an *equational theory* if it is the equational theory of some variety (see [5, Def. 14.9]). A set of identities is an equational theory iff it contains all of its consequences. A set $\Sigma \subseteq \mathrm{Id}(\mathcal{V})$ is an equational base for a variety $\mathcal{V}$ iff all identities in $\mathrm{Id}(\mathcal{V})$ are consequences of the identities in Σ, or equivalently if the smallest equational theory that contains Σ is $\mathrm{Id}(\mathcal{V})$.

We say that a term $t(x_1, \dots, x_n)$ is *constant* in a variety $\mathcal{V}$ if for every algebra $A \in \mathcal{V}$, the term operation $t^A(x_1, \dots, x_n)$ has a constant value, or equivalently if

$$\mathcal{V} \models t(x_1, \dots, x_n) = t(y_1, \dots, y_n).$$

Note that every nullary term is constant.

Lemma 2.3. *A term that is $\mathcal{V}$–equivalent to a constant term of a variety $\mathcal{V}$ is also a constant term of $\mathcal{V}$.*

Proof. The term operations that correspond to $\mathcal{V}$–equivalent terms have the same values in algebras $A \in \mathcal{V}$, so either both have a constant value or both do not. $\square$

Let $\mathcal{V}$ and $\mathcal{V}'$ be varieties of types Ω and Ω' respectively. Let $\varphi\colon \Omega \to T_{\Omega'}(\omega)$ be a function that assigns to a symbol $f(x_1, \dots, x_n) \in \Omega$ a term $\varphi(f)(x_1, \dots, x_n) \in T_{\Omega'}(\omega)$. Then there is a corresponding class function $\varphi^*\colon \mathcal{A}_{\Omega'} \to \mathcal{A}_\Omega$ that assigns to an algebra $A \in \mathcal{A}_{\Omega'}$ an algebra $\varphi^*(A) \in \mathcal{A}_\Omega$ that has the same universe as A, and is such that for every n–ary symbol $f \in \Omega$, the basic operation $f^{\varphi^*(A)}(x_1, \dots, x_n)$ is defined as the term operation $\varphi(f)^A(x_1, \dots, x_n)$. For a symbol $c \in \Omega$ of a constant, $\varphi(c)$ is allowed to be a unary term which is constant in $\mathcal{V}'$. In this case the constant $c^{\varphi^*(A)}$ is defined as the constant value of $\varphi(c)^A(x)$. If $\varphi^*(\mathcal{V}') \subseteq \mathcal{V}$, then φ is called an *interpretation* of $\mathcal{V}$ in $\mathcal{V}'$.

Varieties $\mathcal{V}$ and $\mathcal{V}'$ are *equivalent* if there exist interpretations φ of $\mathcal{V}$ in $\mathcal{V}'$ and ψ of $\mathcal{V}'$ in $\mathcal{V}$, such that $\varphi^*(\psi^*(A)) = A$ for all $A \in \mathcal{V}$ and $\psi^*(\varphi^*(B)) = B$ for all $B \in \mathcal{V}'$ (see [16, Sec. 4.12]). For a given variety we will sometimes want to construct an equivalent variety of a different type. We will use combinations of the following three equivalences.

Example 2.4. Let $\mathcal{V}$ be a variety of a type Ω. For an n–ary $f \in \Omega$, let $\Omega' = \Omega \cup \{f'\}$, where f' is also n–ary. Define a variety $\mathcal{V}'$ of the type Ω' by the identities that define $\mathcal{V}$ and the identity $f'(x_1, \dots, x_n) = f(x_1, \dots, x_n)$. Then varieties $\mathcal{V}$ and $\mathcal{V}'$ are equivalent, because f' can be interpreted in $\mathcal{V}$ as f.

Example 2.5. Let $\mathcal{V}$ be a variety of a type Ω. Let $\Omega' = \Omega \cup \{u\}$, where u is unary. Define a variety $\mathcal{V}'$ of the type Ω' by the identities that define $\mathcal{V}$ and the identity $u(x) = x$. Then varieties $\mathcal{V}$ and $\mathcal{V}'$ are equivalent, because u can be interpreted in $\mathcal{V}$ as the term x.

Example 2.6. Let $\mathcal{V}$ be a variety of a type Ω that contains a symbol of a constant c. Let the type Ω' be the same as the type Ω except with c replaced by a unary symbol $c(x)$. Define a variety $\mathcal{V}'$ of the type Ω' by the identities that define $\mathcal{V}$ except with c replaced by $c(x)$, and by the identity $c(x) = c(y)$. Then varieties $\mathcal{V}$ and $\mathcal{V}'$ are equivalent, because c may be interpreted as $c(x)$ and vice versa.

An identity $u = v$ is *regular* if var(u) = var(v); otherwise it is *irregular*. An irregular identity is *strongly irregular* if it is of the form $t(x, y) = x$ for some binary term $t(x, y)$. E.g. the associative law is regular and the absorption laws are strongly irregular. A variety is *regular* if it satisfies only regular identities; otherwise it is *irregular* (see [21, p. 48]). Every irregular variety satisfies some irregular identity of the form $t(x, y) = u(x)$. An irregular variety is *strongly irregular* if it satisfies some strongly irregular identity (see [21, p. 184]).

For a given type Ω, let $\mathcal{S}_\Omega$ be the variety defined by all regular identities of the type Ω. A type is called *plural* if it has no symbols of constants and it has a symbol of a basic operation of arity at least two (see [21, p. 11]). If Ω is a plural type, then $\mathcal{S}_\Omega$ is the unique variety of the type Ω that is equivalent to the variety $\mathcal{S}$ of all semilattices. Algebras in $\mathcal{S}_\Omega$ are called Ω–*semilattices* (see [21, Ex. 1.5.4]). We will denote $\mathcal{S}_\Omega$ simply by $\mathcal{S}$. The variety $\mathcal{S}$ is idempotent and regular.

2.2 Congruences

A *congruence* θ of an algebra A is an equivalence relation on A that *preserves operations*, i.e. for every n–ary $f \in \Omega$, if $(a_1, b_1), \ldots, (a_n, b_n) \in \theta$, then $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta$ (see [5, Def. 5.1]). Equivalence classes of a congruence are called *congruence classes*. We denote the congruence class of an element $a \in A$ by a/θ. The *quotient* A/θ is an algebra whose universe is the set of congruence classes of θ and in which the value of an n–ary basic operation f on congruence classes $a_1/\theta, \ldots, a_n/\theta$ is the congruence class $f(a_1, \ldots, a_n)/\theta$. A homomorphism

$\operatorname{nat}(\theta)\colon A \to A/\theta$ which maps an element $a \in A$ to its congruence class a/θ is called the *natural homomorphism*.

Congruences of an algebra A form a complete lattice. The meet and the join of a family of congruences θ_i, $i \in I$, are given by

$$\bigwedge_{i\in I} \theta_i = \bigcap_{i\in I} \theta_i, \qquad \bigvee_{i\in I} \theta_i = \bigcup_{n\geq 1} \bigcup_{i_1,\dots,i_n\in I} \theta_{i_1} \circ \theta_{i_2} \circ \cdots \circ \theta_{i_n}.$$

The minimum and maximum congruences are

$$\Delta_A = \{(a,a) \mid a \in A\}, \qquad \nabla_A = A \times A.$$

For a homomorphism $h\colon A \to B$, we denote by $\ker h$ the congruence of A defined by

$$(a,b) \in \ker h \qquad \Longleftrightarrow \qquad h(a) = h(b).$$

Theorem 2.7. [5, Thm. 6.12] *If $h\colon A \to B$ is a surjective homomorphism, then there exists an isomorphism $\varphi\colon A/\ker h \to B$ given by*

$$\varphi(a/\ker h) = h(a), \qquad \forall a \in A.$$

For congruences $\theta \subseteq \psi$ of an algebra A, we denote by ψ/θ the congruence of the quotient algebra A/θ defined by

$$(a/\theta, b/\theta) \in \psi/\theta \qquad \Longleftrightarrow \qquad (a,b) \in \psi.$$

Theorem 2.8. [5, Thm. 6.15] *Let θ and ψ be congruences of an algebra A. If $\theta \subseteq \psi$, then there exists an isomorphism $\varphi\colon (A/\theta)/(\psi/\theta) \to A/\psi$ given by*

$$\varphi\big((a/\theta)/(\psi/\theta)\big) = a/\psi, \qquad \forall a \in A.$$

For elements $a \leq b$ of a lattice L, let $[a,b]$ denote the *interval* $\{x \in L \mid a \leq x \leq b\}$. It is a sublattice of L.

Theorem 2.9. [5, Thm. 6.20] *Let A be an algebra and θ be a congruence of A. There is an isomorphism φ between the interval $[\theta, \nabla_A]$ and the lattice of congruences of A/θ given by*

$$\varphi(\psi) = \psi/\theta, \qquad \forall \psi \supseteq \theta.$$

For congruences $\theta \subseteq \psi$ of an algebra A, if C is a congruence class of ψ/θ, then

$$\bigcup C = \{a \in A \mid a/\theta \in C\}$$

is a congruence class of ψ. The assignment $C \mapsto \bigcup C$ coincides with the isomorphism φ of Theorem 2.8.

Lemma 2.10. *Let $\theta \subseteq \psi$ be congruences of an algebra A and C be a congruence class of ψ/θ. Then C is a subalgebra of A/θ iff $\bigcup C$ is a subalgebra of A.*

Proof. Let $f \in \Omega$ and $a_1/\theta, \ldots, a_n/\theta \in C$. Then

$$f(a_1/\theta, \ldots, a_n/\theta) \in C \iff f(a_1, \ldots, a_n)/\theta \in C \iff f(a_1, \ldots, a_n) \in \bigcup C. \qquad \square$$

We say that congruences θ and ψ *permute* if $\theta \circ \psi = \psi \circ \theta$. If all pairs of congruences of an algebra A permute, then we say that A is *congruence permutable*. A variety $\mathcal{V}$ is called *congruence permutable* if all algebras in $\mathcal{V}$ are congruence permutable. Examples of congruence permutable varieties include varieties of groups, quasigroups, rings, and modules. A ternary term $f(x, y, z)$ is called a *Maltsev term* for a variety $\mathcal{V}$ if $\mathcal{V}$ satisfies the identities $f(x, y, y) = x$ and $f(x, x, y) = y$.

Theorem 2.11. [5, Thm. 12.2] *A variety is congruence permutable iff it has a Maltsev term.*

For congruences θ and ψ of an algebra A and $n \geq 0$, let

$$\theta \circ_n \psi = \begin{cases} \Delta_A, & n = 0, \\ (\theta \circ_{n-1} \psi) \circ \theta, & n \text{ is odd}, \\ (\theta \circ_{n-1} \psi) \circ \psi, & n \text{ is even}. \end{cases}$$

We say that congruences θ and ψ n–permute if $\theta \circ_n \psi = \psi \circ_n \theta$. If all pairs of congruences of an algebra A n–permute, then we say that A is *congruence n–permutable*. A variety $\mathcal{V}$ is called *congruence n–permutable* if all algebras in $\mathcal{V}$ are congruence n–permutable. Note that congruence 2–permutability is just congruence permutability.

Theorem 2.12. [11, Thm. 2] *Let $n \geq 2$. A variety $\mathcal{V}$ is congruence n–permutable iff there are terms $p_1(x, y, z), \ldots, p_{n-1}(x, y, z)$ such that $\mathcal{V}$ satisfies the following identities*

(1) $x = p_1(x, y, y)$,

(2) $p_i(x, x, y) = p_{i+1}(x, y, y)$, *for all* $1 \leq i \leq n-2$,

(3) $p_{n-1}(x, x, y) = y$.

The following lemma characterizes the congruence classes which are subalgebras.

Lemma 2.13. *Let θ be a congruence of an algebra A. A congruence class a/θ is a subalgebra of A iff a/θ is an idempotent of A/θ.*

Proof. Assume that a congruence class a/θ is a subalgebra of A. Then $f(a, \dots, a) \in a/\theta$ for every $f \in \Omega$. Thus

$$f(a/\theta, \dots, a/\theta) = f(a, \dots, a)/\theta = a/\theta$$

for every $f \in \Omega$. Hence a/θ is an idempotent of A/θ.

Now assume that a/θ is an idempotent of A/θ. Then for every n–ary $f \in \Omega$ and every $b_1, \dots, b_n \in a/\theta$ one has

$$f(b_1, \dots, b_n)/\theta = f(b_1/\theta, \dots, b_n/\theta) = f(a/\theta, \dots, a/\theta) = a/\theta.$$

Hence $f(b_1, \dots, b_n) \in a/\theta$, and so a/θ is a subalgebra of A. $\square$

2.3 Replica congruences and free algebras

Let $\mathcal{V}$ be a variety and A be an algebra. A congruence θ of A such that $A/\theta \in \mathcal{V}$ is called a $\mathcal{V}$*–congruence* of A. In every algebra A there exists the minimum $\mathcal{V}$–congruence of A, which is the intersection of all $\mathcal{V}$–congruences of A. It is called the $\mathcal{V}$*–replica congruence* of A. We will denote it by $\varrho_A^{\mathcal{V}}$. The quotient $A/\varrho_A^{\mathcal{V}}$ is called the *replica* of A in $\mathcal{V}$. For varieties $\mathcal{V} \subseteq \mathcal{U}$, one has the inclusion $\varrho_A^{\mathcal{U}} \subseteq \varrho_A^{\mathcal{V}}$ for any algebra A. See [21, Sec. 3.3] for the discussion of replicas and replica congruences.

Lemma 2.14. *Let $h\colon A \to B$ be a homomorphism. If $(a, b) \in \varrho_A^{\mathcal{V}}$, then $(h(a), h(b)) \in \varrho_B^{\mathcal{V}}$.*

Proof. There exists a unique homomorphism $h'\colon A/\varrho_A^{\mathcal{V}} \to B/\varrho_B^{\mathcal{V}}$ such that the following diagram is commutative (see [21, diagram on p. 124]):

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle \mathrm{nat}(\varrho_A^{\mathcal{V}})}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{nat}(\varrho_B^{\mathcal{V}})} \\ A/\varrho_A^{\mathcal{V}} & \xrightarrow{h'} & B/\varrho_B^{\mathcal{V}}. \end{array}$$

Suppose $(a, b) \in \varrho_A^{\mathcal{V}}$. Since a and b have the same image under nat$(\varrho_A^{\mathcal{V}})$, they also have the same image under $h' \circ$ nat$(\varrho_A^{\mathcal{V}})$. Thus a and b have the same image under nat$(\varrho_B^{\mathcal{V}}) \circ h$. It follows that $h(a)$ and $h(b)$ have the same image under nat$(\varrho_B^{\mathcal{V}})$. Hence $(h(a), h(b)) \in \varrho_B^{\mathcal{V}}$. □

Lemma 2.15. *Let $\mathcal{V}_i$, $i \in I$, be varieties, $\mathcal{W}$ be the meet $\bigwedge_{i\in I} \mathcal{V}_i$, and A be an algebra. Then*

$$\varrho_A^{\mathcal{W}} = \bigvee_{i\in I} \varrho_A^{\mathcal{V}_i}.$$

Proof. Let $\theta = \bigvee_{i\in I} \varrho_A^{\mathcal{V}_i}$. By Theorem 2.8, for every $i \in I$,

$$A/\theta \cong (A/\varrho_A^{\mathcal{V}_i})/(\theta/\varrho_A^{\mathcal{V}_i}).$$

Hence A/θ is a quotient of a $\mathcal{V}_i$–algebra for every $i \in I$, so it lies in the intersection $\mathcal{W}$. Thus θ is a $\mathcal{W}$–congruence of A.

Let ψ be a $\mathcal{W}$–congruence of A. Then for every $i \in I$, ψ is also a $\mathcal{V}_i$–congruence, so $\varrho_A^{\mathcal{V}_i} \subseteq \psi$. Hence $\theta \subseteq \psi$, and thus θ is the minimum $\mathcal{W}$–congruence of A. □

Lemma 2.16. *Let $\mathcal{V}$ be a variety, A be an algebra, and θ be a congruence of A. Then*

$$\varrho_{A/\theta}^{\mathcal{V}} = (\theta \vee \varrho_A^{\mathcal{V}})/\theta.$$

Proof. By Theorem 2.8,

$$(A/\theta)/((\theta \vee \varrho_A^{\mathcal{V}})/\theta) \cong A/(\theta \vee \varrho_A^{\mathcal{V}}) \cong (A/\varrho_A^{\mathcal{V}})/((\theta \vee \varrho_A^{\mathcal{V}})/\varrho_A^{\mathcal{V}}).$$

The rightmost algebra is a quotient of the $\mathcal{V}$–replica $A/\varrho_A^{\mathcal{V}}$, so it belongs to $\mathcal{V}$. Hence $(\theta \vee \varrho_A^{\mathcal{V}})/\theta$ is a $\mathcal{V}$–congruence.

By Theorem 2.9, every congruence of A/θ is of the form ψ/θ for some congruence $\psi \supseteq \theta$ of A. Let ψ/θ be a $\mathcal{V}$–congruence of A/θ. Then, by Theorem 2.8,

$$A/\psi \cong (A/\theta)/(\psi/\theta) \in \mathcal{V}.$$

Thus ψ is a $\mathcal{V}$–congruence of A. Hence $\varrho_A^{\mathcal{V}} \subseteq \psi$, which entails $\theta \vee \varrho_A^{\mathcal{V}} \subseteq \psi$. Consequently, one has the inclusion $(\theta \vee \varrho_A^{\mathcal{V}})/\theta \subseteq \psi/\theta$, so $(\theta \vee \varrho_A^{\mathcal{V}})/\theta$ is the minimum $\mathcal{V}$–congruence of A/θ. $\square$

The *free algebra* of a variety $\mathcal{V}$ over a set X of generators is the (unique up to isomorphism) algebra $F_{\mathcal{V}}(X)$ for which there exists an injection $\varphi\colon X \to F_{\mathcal{V}}(X)$ such that for every algebra $A \in \mathcal{V}$, any function $f\colon X \to A$ extends uniquely to a homomorphism $h\colon F_{\mathcal{V}}(X) \to A$, i.e. one has $f = h \circ \varphi$ (see [21, Def. 3.3.4]).

The following two theorems follow from [5, Cor. 10.11] and [5, Cor. 11.10] respectively.

Theorem 2.17. *Every algebra in a variety $\mathcal{V}$ is a homomorphic image of a free algebra of $\mathcal{V}$.*

Theorem 2.18. *Let $\mathcal{P}$ be a prevariety. All free algebras of the variety $\mathsf{H}(\mathcal{P})$ belong to $\mathcal{P}$.*

Corollary 2.19. *A variety contains all of its free algebras.*

Corollary 2.20. *If all free algebras of a variety $\mathcal{V}$ belong to a prevariety $\mathcal{P}$, then $\mathcal{V} \subseteq \mathsf{H}(\mathcal{P})$.*

Proof. By Theorem 2.17, $\mathcal{V} = \mathsf{H}(\{F_{\mathcal{V}}(X) \mid X \text{ is a set}\}) \subseteq \mathsf{H}(\mathcal{P})$. $\square$

A *term algebra* over a set X of variables is an algebra whose universe is the set $T(X)$ of terms and in which the value of an n–ary basic operation f on terms $t_1, \dots, t_n$ is the composite term $f(t_1, \dots, t_n)$ (see [5, Def. 10.4]).

The following theorem follows from [5, Thm. 10.10].

Theorem 2.21. *The free algebra of a variety $\mathcal{V}$ over a set X of generators is given by*

$$F_{\mathcal{V}}(X) = T(X)/\varrho_{T(X)}^{\mathcal{V}}.$$

It follows from [5, Thm. 11.4] that the $\mathcal{V}$–replica congruence of the term algebra $T(X)$ is given by

$$(u, v) \in \varrho_{T(X)}^{\mathcal{V}} \quad \Longleftrightarrow \quad \mathcal{V} \models u = v. \tag{2.1}$$

Elements of $F_{\mathcal{V}}(X)$ are thus equivalence classes that consist of $\mathcal{V}$–equivalent terms. For a term $t \in T(X)$, we will usually denote its equivalence class $t/\varrho_{T(X)}^{\mathcal{V}}$ by $[\,t\,]$.

Corollary 2.22. *Let $\mathcal{V} \subseteq \mathcal{U}$ be varieties and F be the free $\mathcal{U}$–algebra over a set X of generators. Then $\varrho_F^{\mathcal{V}} = \varrho_{T(X)}^{\mathcal{V}} / \varrho_{T(X)}^{\mathcal{U}}$.*

Proof. Since $\varrho_{T(X)}^{\mathcal{U}} \subseteq \varrho_{T(X)}^{\mathcal{V}}$, the join $\varrho_{T(X)}^{\mathcal{U}} \vee \varrho_{T(X)}^{\mathcal{V}}$ coincides with $\varrho_{T(X)}^{\mathcal{V}}$, so the conclusion follows from Lemma 2.16. □

Corollary 2.23. *Let $\mathcal{V} \subseteq \mathcal{U}$ be varieties, F be the free $\mathcal{U}$–algebra over a set X of generators, and $u, v \in T(X)$. Then $\mathcal{V} \models u = v$ iff $([u],[v]) \in \varrho_F^{\mathcal{V}}$.*

Proof. Let T be the term algebra $T(X)$. By Corollary 2.22, $\varrho_F^{\mathcal{V}} = \varrho_T^{\mathcal{V}} / \varrho_T^{\mathcal{U}}$. Hence

$$\mathcal{V} \models u = v \iff (u,v) \in \varrho_T^{\mathcal{V}} \iff (u/\varrho_T^{\mathcal{U}}, v/\varrho_T^{\mathcal{U}}) \in \varrho_T^{\mathcal{V}}/\varrho_T^{\mathcal{U}} \iff ([u],[v]) \in \varrho_F^{\mathcal{V}},$$

where the first equivalence holds by (2.1). □

Corollary 2.24. *Let $\mathcal{V}_i$, $i \in I$, be varieties. If $\bigwedge_{i \in I} \mathcal{V}_i \models u(\boldsymbol{x}) = v(\boldsymbol{x})$, then there exist terms $u = t_1(\boldsymbol{x}), t_2(\boldsymbol{x}), \ldots, t_n(\boldsymbol{x}) = v$ such that for each $1 \leq k < n$, there is some $i \in I$ such that $\mathcal{V}_i \models t_k = t_{k+1}$.*

Proof. Let $\mathcal{W} = \bigwedge_{i \in I} \mathcal{V}_i$. By (2.1), $(u,v) \in \varrho_{T(\boldsymbol{x})}^{\mathcal{W}}$. By Lemma 2.15, $\varrho_{T(\boldsymbol{x})}^{\mathcal{W}} = \bigvee_{i \in I} \varrho_{T(\boldsymbol{x})}^{\mathcal{V}_i}$. Hence there are $t_1, \ldots, t_n \in T(\boldsymbol{x})$ such that $t_1 = u$, $t_n = v$, and for each $1 \leq k < n$, $(t_k, t_{k+1}) \in \varrho_{T(\boldsymbol{x})}^{\mathcal{V}_i}$ for some $i \in I$. The conclusion follows by (2.1). □

2.4 Maltsev products

The definition of the Maltsev product of varieties $\mathcal{V}$ and $\mathcal{W}$ of the same type is slightly simpler than the general definition,

$$\mathcal{V} \circ \mathcal{W} = \{A \mid \exists \theta \;\; A/\theta \in \mathcal{W}, \;\; \forall a \in A \;\; (a/\theta \leq A \;\Rightarrow\; a/\theta \in \mathcal{V})\}. \tag{2.2}$$

Let us mention some of the basic properties of the Maltsev product of varieties (see [15]). For any varieties $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{V}''$, $\mathcal{W}$, $\mathcal{W}'$, and $\mathcal{U}$ of the same type, the following conditions hold:

(1) $\mathcal{V} \circ \mathcal{W}$ is a prevariety,

(2) $\mathcal{V}, \mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W}$,

(3) if $\mathcal{V} \subseteq \mathcal{V}'$ and $\mathcal{W} \subseteq \mathcal{W}'$, then $\mathcal{V} \circ \mathcal{W} \subseteq \mathcal{V}' \circ \mathcal{W}'$,

(4) $\mathcal{V} \circ (\mathcal{V}' \circ \mathcal{V}'') \subseteq (\mathcal{V} \circ \mathcal{V}') \circ \mathcal{V}''$,

(5) $\mathcal{V} \circ \mathcal{T} = \mathcal{V}$,

(6) $\mathcal{A} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{A} = \mathcal{A}$,

(7) if $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$, then $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W} = (\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$.

The following theorem shows that in order to see whether an algebra A belongs to a Maltsev product $\mathcal{V} \circ \mathcal{W}$, we only need to consider the $\mathcal{W}$–replica congruence of A as a candidate for the congruence θ in the definition (2.2). It follows from [15, Cor. 4].

Theorem 2.25. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. Then*

$$\mathcal{V} \circ \mathcal{W} = \{A \mid \forall a \in A \ (a/\varrho_A^{\mathcal{W}} \leq A \ \Rightarrow \ a/\varrho_A^{\mathcal{W}} \in \mathcal{V})\}.$$

We will make use of the following lemma.

Lemma 2.26. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, $A \in \mathcal{V} \circ \mathcal{W}$, and θ be a congruence of A. If $\theta \subseteq \varrho_A^{\mathcal{W}}$, then $A/\theta \in \mathcal{V} \circ \mathcal{W}$.*

Proof. By Lemma 2.16, $\varrho_A^{\mathcal{W}}/\theta$ is the $\mathcal{W}$–replica congruence of A/θ. Let C be a congruence class of $\varrho_A^{\mathcal{W}}/\theta$ which is a subalgebra of A/θ. Then by Lemma 2.10, $D = \bigcup C$ is a congruence class of $\varrho_A^{\mathcal{W}}$ which is a subalgebra of A. By Theorem 2.25, $D \in \mathcal{V}$. Since C is the image of D under the natural homomorphism nat(θ), one has $C \in \mathcal{V}$. Hence $A/\theta \in \mathcal{V} \circ \mathcal{W}$. $\square$

3 Term idempotents

A term t will be called a *term idempotent* of a variety $\mathcal{V}$ (see [19]) if

$$\mathcal{V} \models f(t, \ldots, t) = t, \qquad \forall f \in \Omega.$$

It follows that for a term idempotent t of $\mathcal{V}$,

$$\mathcal{V} \models u(t, \ldots, t) = t, \qquad \forall u \in T(\omega).$$

The following proposition justifies the name.

Proposition 3.1. [19] *A term t is a term idempotent of a variety $\mathcal{V}$ iff the equivalence class $[\,t\,]$ is an idempotent of the free algebra $F_{\mathcal{V}}(\omega)$.*

Proof. Assume that t is a term idempotent of $\mathcal{V}$. Then for any $f \in \Omega$,

$$f([\,t\,], \ldots, [\,t\,]) = [f(t, \ldots, t)] = [\,t\,],$$

so $[\,t\,]$ is an idempotent of $F_{\mathcal{V}}(\omega)$.

Now assume that $[\,t\,]$ is an idempotent of $F_{\mathcal{V}}(\omega)$. Then for any $f \in \Omega$,

$$[f(t, \ldots, t)] = f([\,t\,], \ldots, [\,t\,]) = [\,t\,],$$

so t is a term idempotent of $\mathcal{V}$. $\square$

The usefulness of term idempotents comes from the following property.

Proposition 3.2. [19] *A term $t(\boldsymbol{x})$ is a term idempotent of a variety $\mathcal{V}$ iff for every $A \in \mathcal{V}$ and $\boldsymbol{a} \in A^{\boldsymbol{x}}$, the value $t(\boldsymbol{a})$ is an idempotent of A.*

Proof. Let $t(x_1, \ldots, x_n)$ be a term idempotent of $\mathcal{V}$ and $A \in \mathcal{V}$. If $b = t(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A$, then for any $f \in \Omega$,

$$f(b, \ldots, b) = f(t(a_1, \ldots, a_n), \ldots, t(a_1, \ldots, a_n)) = t(a_1, \ldots, a_n) = b.$$

Thus b is an idempotent of A.

Assume that for every algebra $A \in \mathcal{V}$ and every $\boldsymbol{a} \in A^{\boldsymbol{x}}$, the value $t(\boldsymbol{a})$ is an idempotent of A. In the free algebra $F = F_{\mathcal{V}}(\omega)$, the equivalence class $[\, t\,]$ is the value of the term operation $t^F(x_1, \ldots, x_n)$ on arguments $[x_1], \ldots, [x_n]$, so it is an idempotent of F. Therefore, by Proposition 3.1, t is a term idempotent of $\mathcal{V}$. $\square$

The motivating example of a term idempotent is the term $x \cdot x^{-1}$ in varieties of groups. An example of a term idempotent such that the corresponding term operation in a given algebra does not necessarily have a constant value is the term $x \cdot x^{-1}$ in varieties of inverse semigroups (see [12, Ch. V]). If a variety $\mathcal{V}$ is idempotent, then every term is a term idempotent of $\mathcal{V}$. On the other hand a variety may have no term idempotents, e.g. no term is a term idempotent of $\mathcal{S}g$. Note that a variety is idempotent iff x is its term idempotent.

If a variety $\mathcal{V}$ has term idempotents, then for any algebra A and any $\mathcal{V}$–congruence θ of A, one can characterize the congruence classes of θ which are subalgebras of A as follows.

Proposition 3.3. *Let $\mathcal{V}$ be a variety, $t(\boldsymbol{x})$ be a term idempotent of $\mathcal{V}$, A be an algebra, and θ be a $\mathcal{V}$–congruence of A. A congruence class of θ is a subalgebra of A iff it contains $t(\boldsymbol{a})$ for some $\boldsymbol{a} \in A^{\boldsymbol{x}}$.*

Proof. Let A be an algebra. If a/θ is a subalgebra of A, then it contains the value $t(a, \ldots, a)$. Now assume that a/θ contains a value $t(b_1, \ldots, b_n)$. Then

$$a/\theta = t(b_1, \ldots, b_n)/\theta = t(b_1/\theta, \ldots, b_n/\theta).$$

Thus, by Proposition 3.2, a/θ is an idempotent of A/θ. Hence, by Lemma 2.13, a/θ is a subalgebra of A. $\square$

Corollary 3.4. *If $\mathcal{V}$ is an idempotent variety, then for any algebra A, all congruence classes of any $\mathcal{V}$–congruence of A are subalgebras of A.*

Proof. If $\mathcal{V}$ is an idempotent variety, then the term x is a term idempotent of $\mathcal{V}$. The corresponding term operation is the identity function which has every element of A as its value. $\square$

Corollary 3.5. *Let $\mathcal{V}$ be a variety that has term idempotents, $A \in \mathcal{V}$, and θ be a congruence of A. Then a congruence class of θ is a subalgebra of A iff it contains an idempotent.*

Proof. Let t be any term idempotent of $\mathcal{V}$. Since θ is a $\mathcal{V}$–congruence and each value of $t^A(\boldsymbol{x})$ is an idempotent, the conclusion follows from Proposition 3.3. $\square$

The existence of term idempotents in a variety $\mathcal{V}$, the existence of congruence classes of $\mathcal{V}$–congruences which are subalgebras, and the existence of idempotents in $\mathcal{V}$–algebras are closely related. A part of the following result appears in a different form in [6, Thm. 9].

Proposition 3.6. *Let $\mathcal{V}$ be a variety. The following conditions are equivalent.*

(i) *$\mathcal{V}$ has term idempotents.*

(ii) *Every $\mathcal{V}$–algebra has idempotents.*

(iii) *For any algebra A, every $\mathcal{V}$–congruence of A has a congruence class which is a subalgebra of A.*

Proof. Assume (iii). Then for any $A \in \mathcal{V}$, the minimum congruence Δ_A has a congruence class $\{a\}$ which is a subalgebra. By definition, a is an idempotent of A, so (ii) holds. Assume (ii). Then $F_{\mathcal{V}}(\omega)$ has an idempotent $[\,t\,]$. By Proposition 3.1, t is a term idempotent of $\mathcal{V}$. Hence (i) holds. By Proposition 3.3, (i) implies (iii). $\square$

Let $\mathrm{TI}(\mathcal{V}) \subseteq T(\omega)$ denote the set of term idempotents of a variety $\mathcal{V}$.

Proposition 3.7. *Let $\mathcal{V}$ be a variety of a type Ω. The set $\mathrm{TI}(\mathcal{V})$ has the following properties.*

(1) *If $u \in \mathrm{TI}(\mathcal{V})$ and $\mathcal{V} \models u = v$, then $v \in \mathrm{TI}(\mathcal{V})$.*

(2) *If $t(x_1, \ldots, x_n) \in \mathrm{TI}(\mathcal{V})$, then $t(p_1, \ldots, p_n) \in \mathrm{TI}(\mathcal{V})$ for any terms $p_1, \ldots, p_n$.*

(3) *If $t_1, \ldots, t_n \in \mathrm{TI}(\mathcal{V})$ are pairwise $\mathcal{V}$–equivalent, then $p(t_1, \ldots, t_n) \in \mathrm{TI}(\mathcal{V})$ for any term $p(x_1, \ldots, x_n)$. Furthermore, $\mathcal{V} \models p(t_1, \ldots, t_n) = t_1$.*

(4) *If Ω contains a symbol of arity at least two and $\mathrm{TI}(\mathcal{V})$ contains a term of some positive arity, then $\mathrm{TI}(\mathcal{V})$ contains terms of every positive arity.*

(5) *If $\mathrm{TI}(\mathcal{V})$ is nonempty, then it contains a nullary term or a unary term.*

Proof. (1): If $\mathcal{V} \models u = v$, then $[u] = [v]$ in $F_{\mathcal{V}}(\omega)$, so by Proposition 3.1, either both u and v are term idempotents of $\mathcal{V}$, or none of them is.

(2): For every $f \in \Omega$, $\mathcal{V} \models f(t, \ldots, t) = t$. Thus for every $f \in \Omega$, also

$$\mathcal{V} \models f(t(p_1, \ldots, p_n), \ldots, t(p_1, \ldots, p_n)) = t(p_1, \ldots, p_n).$$

Hence $t(p_1, \ldots, p_n)$ is a term idempotent of $\mathcal{V}$.

(3): The term $p(t_1, \ldots, t_n)$ is $\mathcal{V}$–equivalent to $p(t_1, \ldots, t_1)$, which is $\mathcal{V}$–equivalent to t_1, so by (1), $p(t_1, \ldots, t_n)$ is a term idempotent of $\mathcal{V}$.

(4): Let $n \geq 1$. Let $t(x_1, \ldots, x_k)$ be a term idempotent of a positive arity. In the type Ω there exist terms of any positive arity, so let p be an n–ary term. By (2), the n–ary term $t(p, \ldots, p)$ is a term idempotent of $\mathcal{V}$.

(5): Follows from (4). $\square$

If $\mathcal{V} \subseteq \mathcal{W}$, then TI($\mathcal{W}$) $\subseteq$ TI($\mathcal{V}$). More generally, one has the following relationships.

Proposition 3.8. *Let $\mathcal{V}_i$, $i \in I$, be varieties. Then*

(1) TI($\bigvee_{i \in I} \mathcal{V}_i$) $= \bigcap_{i \in I}$ TI($\mathcal{V}_i$),

(2) $\bigcup_{i \in I}$ TI($\mathcal{V}_i$) $\subseteq$ TI($\bigwedge_{i \in I} \mathcal{V}_i$).

Proof. Since Id($\bigvee_{i \in I} \mathcal{V}_i$) $= \bigcap_{i \in I}$ Id($\mathcal{V}_i$), identities $f(u, \ldots, u) = u$, $f \in \Omega$, are true in $\bigvee_{i \in I} \mathcal{V}_i$ iff for every $i \in I$, they are true in $\mathcal{V}_i$. Therefore (1) follows. For any $j \in I$, $\bigwedge_{i \in I} \mathcal{V}_i \subseteq \mathcal{V}_j$, so TI($\mathcal{V}_j$) $\subseteq$ TI($\bigwedge_{i \in I} \mathcal{V}_i$). Therefore (2) follows. $\square$

The following counterexample shows that the inclusion of Proposition 3.8(2) cannot be replaced by equality.

Counterexample 3.9. Let $\Omega = \{\cdot, \star\}$ be a type with two binary basic operation symbols. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the type Ω defined by the identities $x \cdot x = x$ and $x \star x = x$ respectively. Then $\mathcal{V}$ and $\mathcal{W}$ have no term idempotents, so TI($\mathcal{V}$) $\cup$ TI($\mathcal{W}$) $= \varnothing$, whereas $\mathcal{V} \wedge \mathcal{W}$ is idempotent, so TI($\mathcal{V} \wedge \mathcal{W}$) $= T(\omega)$.

A constant term idempotent $p(x)$ of arity at most one is called a *polar term*. A variety $\mathcal{V}$ that has a polar term is called *polarized*. For an algebra $A \in \mathcal{V}$, the constant value of $p^A(\boldsymbol{x})$ is the unique idempotent of A, which is called the *pole* of A. These notions are due to Maltsev [15]. E.g. every variety of groups is polarized with a polar term $x \cdot x^{-1}$ and the pole of a given group is its identity element. The following proposition summarizes the results proven by Maltsev at the beginning of Section 3 of [15].

Proposition 3.10. *Let $\mathcal{V}$ be a variety. The following conditions are equivalent.*

(i) *$\mathcal{V}$ is polarized.*

(ii) *Every $\mathcal{V}$–algebra has a unique idempotent.*

(iii) *For any algebra A, every $\mathcal{V}$–congruence of A has a unique congruence class which is a subalgebra of A.*

If $t(x_1, \ldots, x_n)$ is a constant term idempotent of a variety $\mathcal{V}$, then $t(x, \ldots, x)$ is a polar term of $\mathcal{V}$. Hence a variety is polarized iff it has a constant term idempotent. We will show that the existence of a term idempotent and the existence of a constant term together imply the existence of a constant term idempotent.

Proposition 3.11. *Let $\mathcal{V}$ be a variety. If u is a term idempotent of $\mathcal{V}$ and v is a constant term of $\mathcal{V}$, then u and v are $\mathcal{V}$–equivalent constant term idempotents of $\mathcal{V}$.*

Proof. The variety $\mathcal{V}$ satisfies the identity $v(x_1, \ldots, x_n) = v(y_1, \ldots, y_n)$. Consequently $\mathcal{V}$ satisfies the identity $v(x_1, \ldots, x_n) = v(u, \ldots, u)$. Since u is a term idempotent of $\mathcal{V}$, the right-hand side of this identity is $\mathcal{V}$–equivalent to u. Hence $\mathcal{V}$ satisfies the identity $v(x_1, \ldots, x_n) = u$. Thus u and v are $\mathcal{V}$–equivalent, so by Proposition 3.7(1) and Lemma 2.3, they are both constant term idempotents of $\mathcal{V}$. $\square$

Corollary 3.12. [19] *All term idempotents of a polarized variety $\mathcal{V}$ are pairwise $\mathcal{V}$–equivalent and constant.*

4 Equational base

By Theorem 1.1, the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of varieties $\mathcal{V}$ and $\mathcal{W}$ is a prevariety, so the variety generated by $\mathcal{V} \circ \mathcal{W}$ coincides with the class $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ of all homomorphic images of algebras in $\mathcal{V} \circ \mathcal{W}$. We will construct an equational base for the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$. For a variety $\mathcal{W}$ and a set Σ of equations, let us define a set $\Sigma^{\mathcal{W}}$ of equations in the following way (see [18]). If $\mathcal{W}$ has no term idempotents, then $\Sigma^{\mathcal{W}} = \varnothing$. Otherwise

$$\Sigma^{\mathcal{W}} = \{u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n) \mid (u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)) \in \Sigma,$$
$$t_1, \ldots, t_n \text{ are pairwise } \mathcal{W}\text{–equivalent term idempotents of } \mathcal{W}\}.$$

We will show that if a variety $\mathcal{V}$ is defined by a set Σ of identities, then the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is defined by $\Sigma^{\mathcal{W}}$.

Lemma 4.1. [18, Lem. 2.2] *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and Σ be a set of identities. If the identities in Σ are true in $\mathcal{V}$, then the identities in $\Sigma^{\mathcal{W}}$ are true in $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$.*

Proof. Let $A \in \mathcal{V} \circ \mathcal{W}$, $u(y_1, \ldots, y_n) = v(y_1, \ldots, y_n)$ be an identity true in $\mathcal{V}$, $t_1(\boldsymbol{x}), \ldots, t_n(\boldsymbol{x})$ be pairwise $\mathcal{W}$–equivalent term idempotents of $\mathcal{W}$, and $\boldsymbol{a} \in A^{\boldsymbol{x}}$. For any $1 \leq i \leq n$,

$$t_i(\boldsymbol{a})/\varrho_A^{\mathcal{W}} = t_i(a_1/\varrho_A^{\mathcal{W}}, \ldots, a_k/\varrho_A^{\mathcal{W}}) = t_1(a_1/\varrho_A^{\mathcal{W}}, \ldots, a_k/\varrho_A^{\mathcal{W}}) = t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}},$$

so elements $t_i(\boldsymbol{a})$, $1 \leq i \leq n$, all lie in the congruence class $t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}}$. Since t_1 is a term idempotent of $\mathcal{W}$, for any $f \in \Omega$,

$$f(t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}}, \ldots, t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}}) = f(t_1(\boldsymbol{a}), \ldots, t_1(\boldsymbol{a}))/\varrho_A^{\mathcal{W}} = t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}}.$$

Thus $t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}}$ is an idempotent of $A/\varrho_A^{\mathcal{W}}$. By Lemma 2.13, $t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}}$ is a subalgebra of A. Since $A \in \mathcal{V} \circ \mathcal{W}$, $t_1(\boldsymbol{a})/\varrho_A^{\mathcal{W}}$ belongs to $\mathcal{V}$, so it satisfies the identity $u = v$. Consequently,

$$u(t_1(\boldsymbol{a}), \ldots, t_n(\boldsymbol{a})) = v(t_1(\boldsymbol{a}), \ldots, t_n(\boldsymbol{a})).$$

Therefore A satisfies the identity $u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n)$.

Let Σ be a set of identities true in $\mathcal{V}$ and let $\mathcal{U}$ be the variety defined by $\Sigma^{\mathcal{W}}$. We have shown that $\mathcal{V} \circ \mathcal{W} \subseteq \mathcal{U}$. It follows that $\mathsf{H}(\mathcal{V} \circ \mathcal{W}) \subseteq \mathcal{U}$, and so every algebra in $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ satisfies the identities in $\Sigma^{\mathcal{W}}$. $\square$

Theorem 4.2. [18, Thm. 2.10] *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If Σ is an equational base for $\mathcal{V}$, then $\Sigma^{\mathcal{W}}$ is an equational base for $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$.*

Proof. Let Σ be an equational base for $\mathcal{V}$ and $\mathcal{U}$ be the variety defined by $\Sigma^{\mathcal{W}}$. By Lemma 4.1, $\mathsf{H}(\mathcal{V} \circ \mathcal{W}) \subseteq \mathcal{U}$. To prove the opposite inclusion, we will show that all free $\mathcal{U}$–algebras belong to $\mathcal{V} \circ \mathcal{W}$. Let X be a set and T be the term algebra $T(X)$. Since

$$\mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W} \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W}) \subseteq \mathcal{U},$$

one has the inclusion $\varrho_T^{\mathcal{U}} \subseteq \varrho_T^{\mathcal{W}}$. By Corollary 2.22, the congruence $\varrho_T^{\mathcal{W}}/\varrho_T^{\mathcal{U}}$ is the $\mathcal{W}$–replica congruence of the free $\mathcal{U}$–algebra $F_{\mathcal{U}}(X) = T(X)/\varrho_T^{\mathcal{U}}$. Let C be a congruence class of $\varrho_T^{\mathcal{W}}/\varrho_T^{\mathcal{U}}$ which is a subalgebra of $F_{\mathcal{U}}(X)$, $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ be some identity from Σ, and $t_1/\varrho_T^{\mathcal{U}}, \ldots, t_n/\varrho_T^{\mathcal{U}} \in C$. Then $\bigcup C$ is a congruence class of $\varrho_T^{\mathcal{W}}$ which, by Lemma 2.10, is a subalgebra of A. Thus, by Lemma 2.13, $\bigcup C$ is an idempotent of $F_{\mathcal{W}}(X)$. Consequently, by Proposition 3.1, $t_1, \ldots, t_n \in \bigcup C$ are pairwise $\mathcal{W}$–equivalent term idempotents of $\mathcal{W}$. Hence the identity $u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n)$ belongs to $\Sigma^{\mathcal{W}}$, and so it is true in $\mathcal{U}$. By (2.1), terms $u(t_1, \ldots, t_n)$ and $v(t_1, \ldots, t_n)$ lie in the same congruence class of $\varrho_T^{\mathcal{U}}$. Therefore

$$u(t_1/\varrho_T^{\mathcal{U}}, \ldots, t_n/\varrho_T^{\mathcal{U}}) = u(t_1, \ldots, t_n)/\varrho_T^{\mathcal{U}} = v(t_1, \ldots, t_n)/\varrho_T^{\mathcal{U}} = v(t_1/\varrho_T^{\mathcal{U}}, \ldots, t_n/\varrho_T^{\mathcal{U}}).$$

Thus C satisfies the identities in Σ, which implies that $C \in \mathcal{V}$. Hence $F_{\mathcal{U}}(X) \in \mathcal{V} \circ \mathcal{W}$. By Corollary 2.20, $\mathcal{U} \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$. We have shown that varieties $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ and $\mathcal{U}$ coincide. $\square$

Corollary 4.3. [18, Cor. 2.11] *If a variety $\mathcal{W}$ has no term idempotents, then $\mathsf{H}(\mathcal{V} \circ \mathcal{W}) = \mathcal{A}$ for any variety $\mathcal{V}$.*

Corollary 4.4. *If a variety $\mathcal{V}$ satisfies an identity $u = v$ such that both u and v are nullary terms, then for any variety $\mathcal{W}$ that has term idempotents, $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ also satisfies $u = v$.*

Example 4.5. The variety $\mathcal{S}$ of Ω–semilattices is idempotent, so every term is a term idempotent of $\mathcal{S}$. Terms t and s are equivalent in $\mathcal{S}$ iff $\text{var}(t) = \text{var}(s)$. Thus for a variety $\mathcal{V}$ defined by a set Σ of identities, the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{S})$ is defined by the identities

$$\Sigma^{\mathcal{S}} = \{u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n) \mid$$

$$(u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)) \in \Sigma, \quad \forall 1 \leq i, j \leq n \quad \text{var}(t_i) = \text{var}(t_j)\}.$$

E.g. $\mathcal{L}z$ is defined by $\Sigma = \{x \cdot y = x\}$, so $\mathsf{H}(\mathcal{L}z \circ \mathcal{S})$ is defined by

$$\Sigma^{\mathcal{S}} = \{p \cdot q = p \mid \text{var}(p) = \text{var}(q)\}.$$

Example 4.6. Let $\mathcal{W}$ be a polarized variety with a polar term $p(x)$. By Corollary 3.12, all term idempotents of $\mathcal{W}$ are $\mathcal{W}$–equivalent, so a term is a term idempotent of $\mathcal{W}$ iff it is $\mathcal{W}$–equivalent to $p(x)$. Thus for a variety $\mathcal{V}$ defined by a set Σ of identities, the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is defined by the identities

$$\Sigma^{\mathcal{W}} = \{u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n) \mid$$

$$(u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)) \in \Sigma, \quad \forall 1 \leq i \leq n \quad \mathcal{W} \models t_i = p(x)\}.$$

For varieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$, the Maltsev $\mathcal{U}$–product $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$ coincides with the intersection $(\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$. For any prevarieties $\mathcal{P}$ and $\mathcal{P}'$, there is an inclusion $\mathsf{H}(\mathcal{P} \cap \mathcal{P}') \subseteq \mathsf{H}(\mathcal{P}) \cap \mathsf{H}(\mathcal{P}')$. Hence $\mathsf{H}(\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}) \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$.

Proposition 4.7. *Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$ be varieties. Then*

$$\mathsf{H}(\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}) = \mathsf{H}(\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}.$$

Proof. Let $\mathcal{M}$ be the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ and T be the term algebra $T(X)$. Then $F = T/\varrho_T^{\mathcal{M} \cap \mathcal{U}}$ is a free algebra of $\mathcal{M} \cap \mathcal{U}$. We will show that $F \in \mathcal{V} \circ \mathcal{W}$. By Lemma 2.15, $\varrho_T^{\mathcal{M} \cap \mathcal{U}} = \varrho_T^{\mathcal{M}} \vee \varrho_T^{\mathcal{U}}$. Let $F' = T/\varrho_T^{\mathcal{M}}$ and let $\theta = (\varrho_T^{\mathcal{M}} \vee \varrho_T^{\mathcal{U}})/\varrho_T^{\mathcal{M}}$. By Theorem 2.8, $F \cong F'/\theta$. By Theorem 2.18, $F' \in \mathcal{V} \circ \mathcal{W}$, because F' is a free algebra of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$. By Lemma 2.16, $\varrho_{F'}^{\mathcal{W}} = (\varrho_T^{\mathcal{M}} \vee \varrho_T^{\mathcal{W}})/\varrho_T^{\mathcal{M}}$. Since $\mathcal{W} \subseteq \mathcal{U}$, $\varrho_T^{\mathcal{U}} \subseteq \varrho_T^{\mathcal{W}}$, so $\theta \subseteq \varrho_{F'}^{\mathcal{W}}$. By Lemma 2.26, $F'/\theta \in \mathcal{V} \circ \mathcal{W}$.

We have shown that all free algebras of $\mathsf{H}(\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$ belong to $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W} = (\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$. Thus, by Corollary 2.20, $\mathsf{H}(\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U} \subseteq \mathsf{H}(\mathcal{V} \circ_{\mathcal{U}} \mathcal{W})$. $\square$

Theorem 4.2 and Proposition 4.7 yield the following result.

Theorem 4.8. *Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$ be varieties. If Σ is an equational base for $\mathcal{V}$, then the variety $\mathsf{H}(\mathcal{V} \circ_{\mathcal{U}} \mathcal{W})$ is defined relative to $\mathcal{U}$ by $\Sigma^{\mathcal{W}}$.*

Iskander [13, Thm. 4.11] provided a set of identities that defines the variety $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$ relative to $\mathcal{U}$ for a weakly congruence permutable variety $\mathcal{U}$. That set of identities also depends on a chosen equational base Σ for $\mathcal{V}$. It forms a subset of $\Sigma^{\mathcal{W}}$, because the term f that occurs in its definition is a term idempotent of $\mathcal{U}$, and thus also a term idempotent of $\mathcal{W}$.

Let us apply Lemma 4.1 to obtain a result on term idempotents of the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$.

Proposition 4.9. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If $t(x_1, \ldots, x_n)$ is a term idempotent of $\mathcal{V}$ and $p_1, \ldots, p_n$ are $\mathcal{W}$–equivalent term idempotents of $\mathcal{W}$, then $t(p_1, \ldots, p_n)$ is a term idempotent of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$.*

Proof. Identities $f(t, \ldots, t) = t$, $f \in \Omega$, are true in $\mathcal{V}$, so by Lemma 4.1, $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ satisfies identities $f(t(p_1, \ldots, p_n), \ldots, t(p_1, \ldots, p_n)) = t(p_1, \ldots, p_n)$, $f \in \Omega$. □

Corollary 4.10. *If varieties $\mathcal{V}$ and $\mathcal{W}$ have term idempotents, then the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has term idempotents.*

Corollary 4.11. *If $\mathcal{W}$ is an idempotent variety, then for any variety $\mathcal{V}$, all unary term idempotents of $\mathcal{V}$ are also term idempotents of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$.*

Proof. Let $t(x)$ be a term idempotent of $\mathcal{V}$. The variable x is a term idempotent of $\mathcal{W}$. The term $t(x)$ is the composition of $t(x)$ and x, so by Proposition 4.9, it is a term idempotent of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$. □

Since $\mathcal{W} \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$, every term idempotent of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is also a term idempotent of $\mathcal{W}$. The following corollary gives a sufficient condition for these two varieties to have the same term idempotents.

Corollary 4.12. *If $\mathcal{V}$ is an idempotent variety and $\mathcal{W}$ is a variety, then the varieties $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ and $\mathcal{W}$ have the same term idempotents.*

Proof. Let t be a term idempotent of $\mathcal{W}$. The substitution of t for the variable x in the term x simply yields t. Since x is a term idempotent of $\mathcal{V}$, by Proposition 4.9, t is a term idempotent of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$. □

Let us examine how the Maltsev product of varieties interacts with regularity, irregularity, and strong irregularity of its factors.

Proposition 4.13. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ have unary term idempotents. Then the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is regular iff at least one of $\mathcal{V}$ and $\mathcal{W}$ is regular.*

Proof. Let $u(x_1, \dots, x_n) = v(x_1, \dots, x_n)$ be an identity true in $\mathcal{V}$. Suppose that $\mathcal{V}$ is regular. Then $u = v$ is regular and thus its consequence $u(t_1, \dots, t_n) = v(t_1, \dots, t_n)$, for any terms $t_1, \dots, t_n$, is regular. Now suppose that $\mathcal{W}$ is regular. Let $t_1, \dots, t_n$ be pairwise $\mathcal{W}$–equivalent terms. Then all terms $t_1, \dots, t_n$ have the same variables, and so the identity $u(t_1, \dots, t_n) = v(t_1, \dots, t_n)$ is regular. Therefore, if at least one of $\mathcal{V}$ and $\mathcal{W}$ is regular, then the identities in the equational base of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ provided by Theorem 4.2 are regular. Since all consequences of regular identities are regular, all identities true in $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ are regular. Hence it is a regular variety.

An irregular variety always satisfies an irregular identity of the form $t(x, y) = u(x)$, where the left-hand side contains the variable y. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are irregular. Then there are irregular identities $t(x, y) = u(x)$ and $s(x, y) = v(x)$ true in $\mathcal{V}$ and $\mathcal{W}$ respectively. Let $h(x)$ be a unary term idempotent of $\mathcal{W}$. The variety $\mathcal{W}$ satisfies the identity $h(s(x, y)) = h(v(x))$ and by Proposition 3.7(2), the terms $h(s(x, y))$ and $h(v(x))$ are term idempotents of $\mathcal{W}$. Thus by Lemma 4.1, $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ satisfies the identity

$$t\big(h(v(x)), h(s(x, y))\big) = u(h(v(x))). \tag{4.1}$$

Since terms $t(x, y)$ and $s(x, y)$ contain the variable y, the left-hand side of (4.1) also contains the variable y. Hence (4.1) is irregular, and so $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is an irregular variety. □

Corollary 4.14. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ have unary term idempotents. Then the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is irregular iff both $\mathcal{V}$ and $\mathcal{W}$ are irregular.*

Proposition 4.15. *Let $\mathcal{V}$ and $\mathcal{W}$ be strongly irregular varieties, and let $\mathcal{W}$ be idempotent. Then the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is strongly irregular.*

Proof. Suppose that varieties $\mathcal{V}$ and $\mathcal{W}$ are strongly irregular and $\mathcal{W}$ is idempotent. Then there are strongly irregular identities $t(x, y) = x$ and $s(x, y) = x$ true in $\mathcal{V}$ and $\mathcal{W}$ respectively. Both $s(x, y)$ and x are term idempotents of $\mathcal{W}$, so by Lemma 4.1, $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ satisfies the identity

$$t(x, s(x, y)) = x. \tag{4.2}$$

Since terms $t(x, y)$ and $s(x, y)$ are binary, the left-hand side of (4.2) is also binary. Hence (4.2) is strongly irregular, and so $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is a strongly irregular variety. □

In [7], a class **F** of varieties of the same type is called *robust* if whenever idempotent varieties $\mathcal{V}$ and $\mathcal{W}$ belong to **F**, then $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ belongs to **F**. The results above imply in particular that the following families are robust.

(1) The class of idempotent regular varieties of a given type that has symbols of non-nullary basic operations.

(2) The class of idempotent strongly irregular varieties of a given type.

An idempotent variety of a type that has a non-nullary basic operation symbol $f(x_1, \ldots, x_n)$, has a unary term idempotent $f(x, \ldots, x)$, so Proposition 4.13 is applicable. If a type Ω has a symbol of a basic operation of arity at least two, then for an idempotent variety of the type Ω, irregularity implies strong irregularity. Hence the class of idempotent irregular varieties of the type Ω coincides with (2).

5 Term idempotent identities

If u and v are term idempotents of a variety $\mathcal{V}$ and an identity $u = v$ is true in $\mathcal{V}$, then we will call $u = v$ a *term idempotent identity* of $\mathcal{V}$ (see [19]). By Proposition 3.7(1), if one side of an identity σ true in $\mathcal{V}$ is a term idempotent of $\mathcal{V}$, then the other side of σ must also be a term idempotent of $\mathcal{V}$, and so σ is a term idempotent identity of $\mathcal{V}$. To motivate this definition let us note that the identities in the equational base for $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ provided by Theorem 4.2 are term idempotent identities of $\mathcal{W}$.

Proposition 5.1. *Let $\mathcal{W}$ be a variety and Σ be a set of equations. Every identity in $\Sigma^{\mathcal{W}}$ is a term idempotent identity of $\mathcal{W}$.*

Proof. Every identity in $\Sigma^{\mathcal{W}}$ is of the form

$$u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n), \tag{5.1}$$

where $t_1, \ldots, t_n$ are $\mathcal{W}$–equivalent term idempotents of $\mathcal{W}$ and $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ is an identity in Σ. By Proposition 3.7(3), both sides of (5.1) are term idempotents of $\mathcal{W}$ that are $\mathcal{W}$–equivalent to t_1. Hence (5.1) is a term idempotent identity of $\mathcal{W}$. □

If the identities in Σ are term idempotent identities of $\mathcal{V}$, then the identities in $\Sigma^{\mathcal{W}}$ are additionally term idempotent identities of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$.

Proposition 5.2. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and Σ be a set of equations. If every identity in Σ is a term idempotent identity of $\mathcal{V}$, then every identity in $\Sigma^{\mathcal{W}}$ is a term idempotent identity of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$.*

Proof. Consider the identity

$$u(t_1, \ldots, t_n) = v(t_1, \ldots, t_n), \tag{5.2}$$

where $t_1, \ldots, t_n$ are $\mathcal{W}$–equivalent term idempotents of $\mathcal{W}$ and $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ is an identity in Σ. Then u is a term idempotent of $\mathcal{V}$. By Proposition 4.9, $u(t_1, \ldots, t_n)$ is a term idempotent of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$, so (5.2) is a term idempotent identity of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$. □

Let $\mathcal{V}$ be a variety and Σ be a set of term idempotent identities of $\mathcal{V}$. Let us investigate whether the nontrivial consequences of Σ are also term idempotent identities of $\mathcal{V}$. It is evident that all nontrivial consequences of Σ inferred by an application of any of the rules (e1)–(e3) of equational logic are term idempotent identities of $\mathcal{V}$. By Proposition 3.7(2), the same is true in case of the rule (e4). However a nontrivial consequence of Σ inferred by an application of the rule (e5) may not be a term idempotent identity of $\mathcal{V}$. E.g. in the variety of groups, the term idempotent identity $x \cdot x^{-1} = y \cdot y^{-1}$ has the consequence $(x \cdot x^{-1}) \cdot z = (y \cdot y^{-1}) \cdot z$, which is not a term idempotent identity. We will characterize varieties in which all nontrivial consequences of any set of term idempotent identities are also term idempotent identities.

A *sink* is a subset S of an algebra A such that

$$f(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n) \in S$$

for every $s \in S$, $f \in \Omega$, $a_1, \ldots, a_n \in A$, and $1 \leq i \leq n$. Note that S is a subalgebra of A.

Proposition 5.3. *Let $\mathcal{V}$ be a variety that has term idempotents. The following conditions are equivalent.*

(i) *Every nontrivial consequence of any set of term idempotent identities of $\mathcal{V}$ is a term idempotent identity of $\mathcal{V}$.*

(ii) *The set* $\mathrm{TI}(\mathcal{V})$ *of term idempotents of $\mathcal{V}$ is a sink of $T(\omega)$.*

(iii) *For every $\mathcal{V}$–algebra A, the set* $\mathrm{I}(A)$ *of idempotents of A is a sink of A.*

Proof. First we will prove the equivalence of conditions (i) and (ii). Assume (i). Let $t \in \mathrm{TI}(\mathcal{V})$, $f \in \Omega$, $p_1, \ldots, p_n \in T(\omega)$, and $1 \leq i \leq n$. The identity $f(t, \ldots, t) = t$ is a nontrivial term idempotent identity of $\mathcal{V}$. It has the nontrivial consequence

$$f(p_1, \ldots, p_{i-1}, f(t, \ldots, t), p_{i+1}, \ldots, p_n) = f(p_1, \ldots, p_{i-1}, t, p_{i+1}, \ldots, p_n). \tag{5.3}$$

By (i), (5.3) is a term idempotent identity of $\mathcal{V}$, so its right-hand side is a term idempotent of $\mathcal{V}$. Thus $\mathrm{TI}(\mathcal{V})$ is a sink of $T(\omega)$.

Now assume (ii). Let $u = v$ be a term idempotent identity of $\mathcal{V}$ and consider its consequence

$$t(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n) = t(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n) \tag{5.4}$$

inferred using the rule (e5). Since u and v are term idempotents of $\mathcal{V}$, by (ii), both sides of (5.4) are also term idempotents of $\mathcal{V}$. Hence (5.4) is a term idempotent identity of $\mathcal{V}$. Taking into account the discussion preceding the statement of this proposition, it follows that every nontrivial consequence of a set of term idempotent identities of $\mathcal{V}$ is a term idempotent identity of $\mathcal{V}$.

We will now prove the equivalence of (ii) and (iii). Assume (ii). Let $A \in \mathcal{V}$, $e \in \mathrm{I}(A)$, $f \in \Omega$, $a_1, \ldots, a_n \in A$, and $1 \leq i \leq n$. Let $t(x) \in \mathrm{TI}(\mathcal{V})$. By (ii),

$$f(x_1, \ldots, x_{i-1}, t(x), x_{i+1}, \ldots, x_n) \in \mathrm{TI}(\mathcal{V}).$$

Since $t(e) = e$,

$$f(a_1, \ldots, a_{i-1}, t(e), a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, e, a_{i+1}, \ldots, a_n),$$

so by Proposition 3.2,

$$f(a_1, \ldots, a_{i-1}, e, a_{i+1}, \ldots, a_n) \in \mathrm{I}(A).$$

Hence $\mathrm{I}(A)$ is a sink of A.

Now assume (iii). Let $t \in \mathrm{TI}(\mathcal{V})$, $f \in \Omega$, $p_1, \ldots, p_n \in T(\omega)$, and $1 \leq i \leq n$. Let F be the free $\mathcal{V}$–algebra $F_{\mathcal{V}}(\omega)$. By Proposition 3.1, $[t] \in \mathrm{I}(F)$. By (iii), $\mathrm{I}(F)$ is a sink of F, so

$$[f(p_1, \ldots, p_{i-1}, t, p_{i+1}, \ldots, p_n)] = f([p_1], \ldots, [p_{i-1}], [t], [p_{i+1}], \ldots, [p_n]) \in \mathrm{I}(F).$$

Thus, by Proposition 3.1,

$$f(p_1, \ldots, p_{i-1}, t, p_{i+1}, \ldots, p_n) \in \mathrm{TI}(\mathcal{V}).$$

Hence $\mathrm{TI}(\mathcal{V})$ is a sink of $T(\omega)$. $\square$

An element z of an algebra A is called a *zero* of A if it forms a one-element sink $\{z\}$. If a zero exists in an algebra, then it is unique. A constant term $p(x)$ of a variety $\mathcal{V}$ will be called a *zero term* of $\mathcal{V}$ if for every algebra $A \in \mathcal{V}$, the unique value of $p^A(x)$ is the zero of A (see [19]). Equivalently, a term $p(x)$ is a zero term of a variety $\mathcal{V}$ if it is constant and

$$\mathcal{V} \models f(y_1, \ldots, y_{i-1}, p(x), y_{i+1}, \ldots, y_n) = p(x), \qquad \forall f \in \Omega \ \forall 1 \leq i \leq n. \tag{5.5}$$

Note that a zero term is a polar term. Thus a variety $\mathcal{V}$ that has a zero term is necessarily polarized and the pole of every algebra $A \in \mathcal{V}$ is the zero of A.

Proposition 5.4. [19, Prop. 6.4] *Let $\mathcal{V}$ be a polarized variety. The following conditions are equivalent.*

(i) *Every nontrivial consequence of any set of term idempotent identities of $\mathcal{V}$ is a term idempotent identity of $\mathcal{V}$.*

(ii) *The pole of any $\mathcal{V}$–algebra A is a zero of A.*

(iii) *Polar terms of $\mathcal{V}$ are zero terms of $\mathcal{V}$.*

(iv) *$\mathcal{V}$ has a zero term.*

Proof. Every $\mathcal{V}$–algebra has a unique idempotent – its pole. Hence, by Proposition 5.3, the conditions (i) and (ii) are equivalent. The conditions (ii) and (iii) are equivalent by the definition of a zero term. Clearly, (iii) implies (iv). Assume (iv). Let $p(x)$ be a zero term of $\mathcal{V}$. By Corollary 3.12, every polar term of $\mathcal{V}$ is $\mathcal{V}$–equivalent to $p(x)$. It follows that every polar term of $\mathcal{V}$ satisfies the identities (5.5). Thus (iii) holds. $\square$

6 Term idempotent varieties

We will say that a variety $\mathcal{V}$ is *term idempotent* if every nontrivial identity true in $\mathcal{V}$ is a term idempotent identity of $\mathcal{V}$ (see [19]). In the previous chapter we saw that a set of term idempotent identities may have consequences that are not term idempotent identities. Hence for a variety to be term idempotent, it is not sufficient that it has an equational base which consists of term idempotent identities. As a corollary of Proposition 5.3, one obtains the following characterization of term idempotent varieties.

Proposition 6.1. *A variety $\mathcal{V}$ is term idempotent iff it has an equational base that consists of term idempotent identities and either of the following equivalent conditions is satisfied.*

(i) *The set* $\mathrm{TI}(\mathcal{V})$ *of term idempotents of $\mathcal{V}$ is a sink of $T(\omega)$.*

(ii) *For every $\mathcal{V}$–algebra A, the set* $\mathrm{I}(A)$ *of idempotents of A is a sink of A.*

If $\mathcal{V}$ is an idempotent variety, then every term is a term idempotent of $\mathcal{V}$. It follows that all identities true in $\mathcal{V}$ are term idempotent identities of $\mathcal{V}$. Hence every idempotent variety is a term idempotent variety. Let us look at some examples of term idempotent varieties that are not idempotent.

Example 6.2. The variety $\mathcal{A}$ satisfies only trivial identities, so it is a term idempotent variety. It is the only term idempotent variety that has no term idempotents.

Example 6.3. [19, Ex. 3.5] Recall that $\mathcal{S}g$ is the variety of semigroups and $\mathcal{R}b$ is the variety of rectangular bands. Let $\mathcal{R}s$ be the subvariety of $\mathcal{S}g$ defined relative to $\mathcal{S}g$ by the identity

$$(x \cdot y) \cdot z = x \cdot z. \tag{6.1}$$

Note that $\mathcal{R}b$ is the subvariety of $\mathcal{R}s$ defined relative to $\mathcal{R}s$ by the idempotent law $x \cdot x = x$. In $\mathcal{S}g$, every term is a product of variables (we will omit the parentheses). If $n \geq 2$ and t is the term $x_1 \cdot \ldots \cdot x_n$, then $\mathcal{R}s \models t = x_1 \cdot x_n$ and

$$\mathcal{R}s \models t \cdot t = x_1 \cdot x_n \cdot x_1 \cdot x_n = x_1 \cdot x_n = t.$$

Consequently, the set of term idempotents of $\mathcal{R}s$ consists of all terms distinct from variables. It is clearly a sink of $T(\omega)$. The identities that define $\mathcal{S}g$ and the identity (6.1) form an equational base for $\mathcal{R}s$ which consists of term idempotent identities. Thus, by Proposition 6.1, $\mathcal{R}s$ is a term idempotent variety.

Example 6.4. [19, Ex. 3.4] We will say that an algebra is *constant* if one of its elements is the constant value of every basic operation. The variety $\mathcal{C}_\Omega$ of all constant algebras of a type Ω is defined by the identities

$$f(x_1, \dots, x_n) = g(y_1, \dots, y_m), \qquad \forall f, g \in \Omega.$$

We will usually omit the subscript and write simply $\mathcal{C}$. The variety $\mathcal{C}$ satisfies a nontrivial identity $u = v$ iff neither u nor v is a variable. Thus, if t is a term distinct from a variable, then $\mathcal{C}$ satisfies the identities $f(t, \dots, t) = t$, for all $f \in \Omega$, so t is a term idempotent of $\mathcal{C}$. Hence every nontrivial identity true in $\mathcal{C}$ is a term idempotent identity. It follows that $\mathcal{C}$ is a term idempotent variety. Note that every term that is not a variable is constant in $\mathcal{C}$.

Of special interest is the variety $\mathcal{C}_{\{\cdot\}}$ for the magma type $\{\cdot\}$. It is defined by the identity $x \cdot y = z \cdot t$. In particular, algebras in $\mathcal{C}_{\{\cdot\}}$ are semigroups. We will denote the variety $\mathcal{C}_{\{\cdot\}}$ of constant semigroups by $\mathcal{C}s$.

Example 6.5. Let Ω be a type. For any natural number n, let T_n be the set of all terms of the type Ω that are distinct from variables and that contain at least n occurrences of variables. Let $\mathcal{C}_n$ be the variety of the type Ω defined by the set of identities $\Sigma_n = \{u = v \mid u, v \in T_n\}$. The rules (e2)–(e5) applied to Σ_n cannot produce an identity that has less than n occurrences of variables on either side, so Σ_n contains all nontrivial identities true in $\mathcal{C}_n$. If $u \in T_n$, then the identities $f(u, \dots, u) = u$, $f \in \Omega$, belong to Σ_n. Therefore every identity in Σ_n is a term idempotent identity of $\mathcal{C}_n$. Hence $\mathcal{C}_n$ is a term idempotent variety. Note that $\mathcal{C}_0$ coincides with $\mathcal{C}$.

In case when Ω is the magma type $\{\cdot\}$, there are no terms distinct from variables that contain less than two occurrences of a variable, so the varieties $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$ coincide with $\mathcal{C}s$. The associative law belongs to T_n iff $n \leq 3$. Hence $\mathcal{C}_3$ is also a variety of semigroups.

Example 6.6. [19, Ex. 3.6] Let $\Omega = \{f\}$ be a type with a single unary basic operation symbol. For a natural number n, let $\mathcal{U}_n$ be the variety of the type Ω defined by the identity

$$f(f^n(x)) = f^n(x). \tag{6.2}$$

The set of term idempotents of $\mathcal{U}_n$ consists of all terms $f^m(x)$, $m \geq n$. It is a sink of $T(\omega)$. The defining identity (6.2) is a term idempotent identity. Hence $\mathcal{U}_n$ is a term idempotent variety.

The varieties in Examples 6.3, 6.4, and 6.5 are irregular, but not strongly irregular. We will show that there are no term idempotent varieties that are strongly irregular and not idempotent.

Proposition 6.7. [19, Prop. 3.8] *If a term idempotent variety is strongly irregular, then it is idempotent.*

Proof. A strongly irregular term idempotent variety $\mathcal{V}$ satisfies a strongly irregular identity $t(x, y) = x$. This identity is nontrivial, so it is a term idempotent identity, and thus its right-hand side x is a term idempotent of $\mathcal{V}$. Therefore $\mathcal{V}$ is idempotent. □

The inclusion of Proposition 3.8(2) can be replaced by equality in case when $\mathcal{V}_i$, $i \in I$, are term idempotent varieties.

Proposition 6.8. *If $\mathcal{V}_i$, $i \in I$, are term idempotent varieties, then* $\mathrm{TI}(\bigwedge_{i\in I} \mathcal{V}_i) = \bigcup_{i\in I} \mathrm{TI}(\mathcal{V}_i)$.

Proof. By Proposition 3.8(2), $\bigcup_{i\in I} \mathrm{TI}(\mathcal{V}_i) \subseteq \mathrm{TI}(\bigwedge_{i\in I} \mathcal{V}_i)$. We will show that the opposite inclusion also holds. Let $\mathcal{W} = \bigwedge_{i\in I} \mathcal{V}_i$. By Theorem 6.10, $\mathcal{W}$ is term idempotent. Let $u \in \mathrm{TI}(\mathcal{W})$. For any $f \in \Omega$, the variety $\mathcal{W}$ satisfies the nontrivial identity $u = f(u, \ldots, u)$. Therefore, by Corollary 2.24, there exists $j \in I$ and a term v distinct from u, such that $\mathcal{V}_j \models u = v$. Since $\mathcal{V}_j$ is term idempotent, $u = v$ is a term idempotent identity of $\mathcal{V}_j$. Hence

$$u \in \mathrm{TI}(\mathcal{V}_j) \subseteq \bigcup_{i\in I} \mathrm{TI}(\mathcal{V}_i). \qquad \square$$

We will now investigate the properties of the class of term idempotent varieties of a given type. The following counterexample shows that this class is not in general closed under subvarieties.

Counterexample 6.9. Let $\mathcal{V}$ be a variety of the type $\Omega = \{f, g, h\}$ with three unary basic operation symbols, defined by the identities

$$f(f(x)) = f(x), \quad g(f(x)) = f(x), \quad h(f(x)) = f(x). \tag{6.3}$$

The non-variable terms of the type Ω are of the form $p_n(\cdots p_2(p_1(x))\cdots)$, where each p_i is f, g, or h. It is easy to see that if σ is a nontrivial consequence of the identities (6.3), then both sides of σ contain at least one occurrence of f. Since $f(x)$ is a term idempotent of $\mathcal{V}$, by Proposition 3.7(2,3), every term containing at least one occurrence of f is a term idempotent of $\mathcal{V}$. Hence $\mathcal{V}$ is a term idempotent variety. Now let $\mathcal{W}$ be the subvariety of $\mathcal{V}$ defined relative to $\mathcal{V}$ by the identity

$$g(x) = h(x). \tag{6.4}$$

It is easy to see that no nontrivial consequence of (6.4) (other than $h(x) = g(x)$) has $g(x)$ as one of its sides. Thus the identity $f(g(x)) = g(x)$ is not true in $\mathcal{W}$, so $g(x)$ is not a term idempotent of $\mathcal{W}$. Hence (6.4) is not a term idempotent identity of $\mathcal{W}$, which implies that $\mathcal{W}$ is not term idempotent.

Theorem 6.10. *The class of term idempotent varieties of a type Ω forms a complete sublattice of the lattice of varieties of the type Ω.*

Proof. The trivial variety and the variety of all algebras, i.e. the smallest and the largest variety respectively, are both term idempotent. Let $\mathcal{V}_i$, $i \in I$, be term idempotent varieties. Let $u = v$ be a nontrivial identity true in the join $\mathcal{U} = \bigvee_{i\in I} \mathcal{V}_i$. Then $u = v$ is true in every $\mathcal{V}_i$, $i \in I$, so it is a term idempotent identity in every $\mathcal{V}_i$, $i \in I$. Hence the identities $f(u, \ldots, u) = u$, $f \in \Omega$, are true in each $\mathcal{V}_i$, $i \in I$, and thus they are also true in $\mathcal{U}$. Therefore $u = v$ is a term idempotent identity of $\mathcal{U}$. Consequently $\mathcal{U}$ is a term idempotent variety.

Let $u = v$ be a nontrivial identity true in the meet $\mathcal{W} = \bigwedge_{i\in I} \mathcal{V}_i$. Then, by Corollary 2.24, there exist $j \in I$ and a term t distinct from u such that $\mathcal{V}_j \models u = t$. Since $\mathcal{V}_j$ is term idempotent, $u = t$ is a term idempotent identity of $\mathcal{V}_j$. Hence u is a term idempotent of $\mathcal{V}_j$, and thus also of $\mathcal{W}$. It follows that $u = v$ is a term idempotent identity of $\mathcal{W}$, so $\mathcal{W}$ is a term idempotent variety. $\square$

By Theorem 6.10, for any variety $\mathcal{V}$, there exist the smallest term idempotent variety that contains $\mathcal{V}$ and the largest term idempotent subvariety of $\mathcal{V}$. We will denote these term idempotent varieties by $\mathcal{V}^{\vartriangle}$ and $\mathcal{V}^{\triangledown}$ respectively. The assignments $\mathcal{V} \mapsto \mathcal{V}^{\vartriangle}$ and $\mathcal{V} \mapsto \mathcal{V}^{\triangledown}$ define respectively a closure operator and a kernel operator on the lattice of all varieties of a given type.

Proposition 6.11. *For a variety $\mathcal{V}$, the variety $\mathcal{V}^{\vartriangle}$ is defined by the identities*

$$\{u = v \mid \mathcal{V} \models u = v,\ \forall t(y, \boldsymbol{x}) \in T(\omega)\ \ t(u, \boldsymbol{x}) \in \mathrm{TI}(\mathcal{V})\}. \tag{6.5}$$

Proof. Let

$$U = \{u \in \mathrm{TI}(\mathcal{V}) \mid \forall t(y, \boldsymbol{x}) \in T(\omega)\ \ t(u, \boldsymbol{x}) \in \mathrm{TI}(\mathcal{V})\}.$$

It is the largest subset of $\mathrm{TI}(\mathcal{V})$ which is a sink. Let Σ be the set (6.5). It consists of all term idempotent identities of $\mathcal{V}$ whose both sides belong to U. By Proposition 6.1, a variety defined by term idempotent identities is term idempotent iff its term idempotents form a sink. Hence the variety $\mathcal{U}$ defined by Σ is a term idempotent variety that contains $\mathcal{V}$. Furthermore, for every other term idempotent variety $\mathcal{U}'$ that contains $\mathcal{V}$, the set $\mathrm{TI}(\mathcal{U}')$ is a sink, so $\mathrm{TI}(\mathcal{U}') \subseteq U$. Thus $\mathrm{Id}(\mathcal{U}') \subseteq \Sigma$. Consequently, $\mathcal{U} \subseteq \mathcal{U}'$. $\square$

Example 6.12. Let $\mathcal{V}$ be a variety that has no term idempotents. Then the set (6.5) is empty, so $\mathcal{V}^{\vartriangle}$ coincides with the variety $\mathcal{A}$ of all algebras. E.g. $\mathcal{S}g^{\vartriangle} = \mathcal{A}$.

Example 6.13. If $\mathrm{TI}(\mathcal{V})$ is a sink of $T(\omega)$, then $\mathcal{V}^{\vartriangle}$ is defined by all term idempotent identities true in $\mathcal{V}$. E.g. if $\mathcal{V}$ and $\mathcal{W}$ are the varieties defined in Counterexample 6.9, then $\mathcal{W}^{\vartriangle} = \mathcal{V}$.

For a variety $\mathcal{V}$, let $\mathrm{S}(\mathcal{V})$ denote the set of sides of all nontrivial identities true in $\mathcal{V}$. E.g. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ is a nontrivial identity true in the variety $\mathcal{S}g$ of semigroups, so both terms $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ belong to $\mathrm{S}(\mathcal{S}g)$.

Proposition 6.14. *For a variety $\mathcal{V}$, the subvariety $\mathcal{V}^{\triangledown}$ is defined relative to $\mathcal{V}$ by the identities*

$$\{f(u, \dots, u) = u \mid u \in \mathrm{S}(\mathcal{V}),\ f \in \Omega\}. \tag{6.6}$$

Proof. If $u \in \mathrm{S}(\mathcal{V})$ is a left-hand side of a nontrivial identity σ true in $\mathcal{V}$, then the terms

$$f(p_1, \dots, p_{i-1}, u, p_{i+1}, \dots, p_n), \quad f \in \Omega,\ p_1, \dots, p_n \in T(\omega),\ 1 \le i \le n, \tag{6.7}$$

are the left-hand sides of the consequences of σ inferred by the rule (e5) of equational logic. Hence the terms (6.7) belong to $\mathrm{S}(\mathcal{V})$, which implies that $\mathrm{S}(\mathcal{V})$ is a sink of $T(\omega)$.

Let Σ be the union of $\mathrm{Id}(\mathcal{V})$ and the set (6.6), and let $\mathcal{W}$ be the variety defined by Σ. Since $\mathrm{TI}(\mathcal{W})$ coincides with $\mathrm{S}(\mathcal{V})$, it is a sink of $T(\omega)$. The identities of Σ are term idempotent identities of $\mathcal{W}$, so by Proposition 6.1, $\mathcal{W}$ is term idempotent. Each term idempotent subvariety of $\mathcal{V}$ satisfies the identities of Σ, so it is a subvariety of $\mathcal{W}$. $\square$

The set of identities provided by Proposition 6.14 is infinite. However sometimes it is possible to find a finite set of identities that defines $\mathcal{V}^{\triangledown}$ relative to $\mathcal{V}$.

Example 6.15. We will show that $\mathcal{S}g^{\triangledown}$ is defined relative to $\mathcal{S}g$ by the identity

$$((x \cdot y) \cdot z) \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z. \tag{6.8}$$

Let $\mathcal{V}$ be the subvariety of $\mathcal{S}g$ defined relative to $\mathcal{S}g$ by the identity (6.8). Then the terms $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ are term idempotents of $\mathcal{V}$. Both sides of any nontrivial identity true in $\mathcal{V}$ contain at least three occurrences of a variable. Hence a term t that is a side of a nontrivial identity true in $\mathcal{V}$ is of the form $(p \cdot q) \cdot r$ or $p \cdot (q \cdot r)$ for some terms p, q, and r. Thus, by Proposition 3.7(2), t is a term idempotent of $\mathcal{V}$. It follows that $\mathcal{V}$ is term idempotent. Since every term idempotent subvariety of $\mathcal{S}g$ must satisfy the identity (6.8), $\mathcal{V}$ coincides with $\mathcal{S}g^{\triangledown}$.

Example 6.16. Let $\mathcal{C}om$ be the variety of commutative magmas. It is defined by the identity $x \cdot y = y \cdot x$. The variety $\mathcal{C}om^{\triangledown}$ is defined relative to $\mathcal{C}om$ by the identity $(x \cdot y) \cdot (x \cdot y) = x \cdot y$. The proof is analogous to that of Example 6.15.

The *regularization* of a variety $\mathcal{V}$ is the variety defined by all regular identities true in $\mathcal{V}$ (see [21, p. 48]). The following proposition shows that the class of term idempotent varieties of a given type is closed under regularization.

Proposition 6.17. [19, Prop. 3.7] *If $\mathcal{V}$ is a term idempotent variety, then the regularization of $\mathcal{V}$ is a term idempotent variety.*

Proof. Let $\widetilde{\mathcal{V}}$ be the regularization of a term idempotent variety $\mathcal{V}$ and let $u = v$ be a nontrivial identity true in $\widetilde{\mathcal{V}}$. Then $u = v$ is also true in $\mathcal{V}$, so u is a term idempotent of $\mathcal{V}$. Thus $\mathcal{V}$ satisfies the identities $f(u, \dots, u) = u$, for all $f \in \Omega$. Since these identities are regular, they are also true in $\widetilde{\mathcal{V}}$. Therefore u is a term idempotent of $\widetilde{\mathcal{V}}$. Hence $\widetilde{\mathcal{V}}$ is a term idempotent variety. □

It is known that for idempotent varieties $\mathcal{V}$ and $\mathcal{W}$, the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is also idempotent. Let us prove this fact using the results of this work. Let $\mathcal{V}$ and $\mathcal{W}$ be idempotent varieties. Then $\mathcal{V}$ satisfies the identities $f(x, \dots, x) = x$, $f \in \Omega$, and the term x is a term idempotent of $\mathcal{W}$. Thus, by Lemma 4.1, $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ satisfies the identities $f(x, \dots, x) = x$, $f \in \Omega$, so it is idempotent. Now let us see when the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is term idempotent.

Proposition 6.18. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If $\mathcal{V}$ is idempotent and* $\mathsf{TI}(\mathcal{W})$ *is a sink of $T(\omega)$, then the variety* $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ *is term idempotent.*

Proof. Let Σ be an equational base for $\mathcal{V}$ and let Ψ be the smallest equational theory that contains $\Sigma^{\mathcal{W}}$. By Theorem 4.2, Ψ is the equational theory of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$. By Proposition 5.1, the identities in $\Sigma^{\mathcal{W}}$ are term idempotent identities of $\mathcal{W}$. Thus, by Proposition 5.3, all nontrivial identities in Ψ are term idempotent identities of $\mathcal{W}$. By Corollary 4.12, varieties $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ and $\mathcal{W}$ have the same term idempotents, so all nontrivial identities in Ψ are also term idempotent identities of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$. □

Corollary 6.19. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If $\mathcal{V}$ is idempotent and $\mathcal{W}$ is term idempotent, then the variety* $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ *is term idempotent.*

One might hope to extend this result to the case when $\mathcal{V}$ is term idempotent. However this is not possible as the following counterexample shows.

Counterexample 6.20. The term idempotent variety $\mathcal{C}s$ of all constant semigroups is defined by the identity $x \cdot y = z \cdot t$, which we will denote by σ. The set $\{\sigma\}^{\mathcal{C}s}$ is thus an equational base for $\mathsf{H}(\mathcal{C}s \circ \mathcal{C}s)$. Every identity in $\{\sigma\}^{\mathcal{C}s}$ is of the form $t_1 \cdot t_2 = t_3 \cdot t_4$ for some terms t_1, t_2, t_3, and t_4 containing at least two occurrences of variables. Therefore a term of the form $t \cdot x$ never occurs as a side of an identity in $\{\sigma\}^{\mathcal{C}s}$. Hence it can occur as a side of a nontrivial identity true in $\mathsf{H}(\mathcal{C}s \circ \mathcal{C}s)$ only as a result of application of the rule (e5), so only in case of an identity of the form $t \cdot x = s \cdot x$. It follows that none of the identities $(t \cdot x) \cdot (t \cdot x) = t \cdot x$, $t \in T(\omega)$, is true in $\mathsf{H}(\mathcal{C}s \circ \mathcal{C}s)$. Consequently, none of the terms $t \cdot x$, $t \in T(\omega)$, is a term idempotent of $\mathsf{H}(\mathcal{C}s \circ \mathcal{C}s)$. Thus $\mathsf{H}(\mathcal{C}s \circ \mathcal{C}s)$ is not term idempotent.

As a corollary of Proposition 5.4 one obtains the following characterization of polarized term idempotent varieties.

Proposition 6.21. *A variety $\mathcal{V}$ is polarized and term idempotent iff it has an equational base that consists of term idempotent identities and either of the following equivalent conditions is satisfied.*

(i) *The pole of any $\mathcal{V}$–algebra A is a zero of A.*

(ii) *Polar terms of $\mathcal{V}$ are zero terms of $\mathcal{V}$.*

(iii) *$\mathcal{V}$ has a zero term.*

Example 6.22. Example 6.5 introduced term idempotent varieties $\mathcal{C}_n$, $n \geq 0$. Let $u(x) \in T_n$. Then $u(x)$ is a term idempotent of $\mathcal{C}_n$. The identity $u(x) = u(y)$ is in Σ_n, so $u(x)$ is a polar term of $\mathcal{C}_n$. Thus $\mathcal{C}_n$ is a polarized variety. This example is a generalization of [19, Ex. 6.2].

Proposition 6.23. *The class of polarized term idempotent varieties of a type Ω together with the variety $\mathcal{A}$ of all algebras of the type Ω forms a complete sublattice of the lattice of term idempotent varieties of the type Ω.*

Proof. Let $\mathcal{V}_i$, $i \in I$, be a family of polarized term idempotent varieties. A subvariety of a polarized variety is also polarized, so the meet $\bigwedge_{i\in I} \mathcal{V}_i$ is a polarized term idempotent variety. Let $\mathcal{U}$ be the join $\bigvee_{i\in I} \mathcal{V}_i$. Since $\mathcal{U}$ is a term idempotent variety, if $\mathrm{TI}(\mathcal{U})$ is empty, then $\mathcal{U}$ coincides with $\mathcal{A}$. Otherwise there is some $p(x) \in \mathrm{TI}(\mathcal{U})$. By Proposition 3.8(1), $\mathrm{TI}(\mathcal{U}) = \bigcap_{i\in I} \mathrm{TI}(\mathcal{V}_i)$. Thus for any $i \in I$, $p(x) \in \mathrm{TI}(\mathcal{V}_i)$, so by Corollary 3.12, $p(x)$ is a polar term of $\mathcal{V}_i$. Hence $p(x)$ is a polar term of $\mathcal{U}$, which implies that $\mathcal{U}$ is a polarized term idempotent variety. □

By Theorem 6.10 and Proposition 6.23, for any variety $\mathcal{V}$ distinct from $\mathcal{A}$, there exists the largest polarized term idempotent subvariety of $\mathcal{V}$. We will denote it by $\mathcal{V}^{\mathrm{pol}}$. Observe that $\mathcal{V}^{\mathrm{pol}} = (\mathcal{V}^{\triangledown})^{\mathrm{pol}}$.

Proposition 6.24. *For a term idempotent variety $\mathcal{V}$ distinct from $\mathcal{A}$, the subvariety $\mathcal{V}^{\mathrm{pol}}$ is defined by the identities*

$$\{u = v \mid u, v \in \mathrm{TI}(\mathcal{V})\}. \tag{6.9}$$

Proof. Let Σ be the set (6.9) and $\mathcal{W}$ be the variety defined by Σ. By Corollary 3.12, all term idempotents of $\mathcal{V}^{\mathrm{pol}}$ are equivalent in $\mathcal{V}^{\mathrm{pol}}$. Thus $\mathcal{V}^{\mathrm{pol}}$ satisfies the identities of Σ, so $\mathcal{V}^{\mathrm{pol}} \subseteq \mathcal{W}$.

Since Σ contains all nontrivial identities true in $\mathcal{V}$, $\mathcal{W}$ is a subvariety of $\mathcal{V}$. Let $p(x)$ be a term idempotent of $\mathcal{V}$. Then $p(x)$ is a polar term of $\mathcal{W}$, because Σ contains the identity $p(x) = p(y)$. By Proposition 6.1, $\mathrm{TI}(\mathcal{V})$ is a sink of $T(\omega)$. Therefore for any n–ary basic operation f and any $1 \le i \le n$,

$$f(y_1, \ldots, y_{i-1}, p(x), y_{i+1}, \ldots, y_n) \in \mathrm{TI}(\mathcal{V}),$$

which implies that

$$\big(f(y_1, \ldots, y_{i-1}, p(x), y_{i+1}, \ldots, y_n) = p(x)\big) \in \Sigma.$$

Hence $p(x)$ is a zero term of $\mathcal{W}$. The equational base Σ for $\mathcal{W}$ consists of term idempotent identities of $\mathcal{W}$. By Proposition 6.21, $\mathcal{W}$ is term idempotent. We have shown that $\mathcal{W}$ is a polarized term idempotent subvariety of $\mathcal{V}$, so $\mathcal{W} \subseteq \mathcal{V}^{\mathrm{pol}}$. □

Proposition 6.25. *For a variety $\mathcal{V}$ distinct from $\mathcal{A}$, the subvariety $\mathcal{V}^{\mathrm{pol}}$ is defined by the identities*

$$\{u = v \mid u, v \in \mathrm{S}(\mathcal{V})\}.$$

Proof. By Proposition 6.24, $\mathcal{V}^{\mathrm{pol}} = (\mathcal{V}^{\triangledown})^{\mathrm{pol}}$ is defined by the identities $\{u = v \mid u, v \in \mathrm{TI}(\mathcal{V}^{\triangledown})\}$. By Proposition 6.14, $\mathrm{TI}(\mathcal{V}^{\triangledown}) = \mathrm{S}(\mathcal{V})$. □

Example 6.26. In the variety $\mathcal{R}s$ of Example 6.3, every term distinct from a variable is a term idempotent. Hence $\mathrm{TI}(\mathcal{R}s)$ coincides with the set T_2 of Example 6.5, and the equational base $\{u = v \mid u, v \in \mathrm{TI}(\mathcal{R}s)\}$ for $\mathcal{R}s^{\mathrm{pol}}$ coincides with the set Σ_2. Consequently $\mathcal{R}s^{\mathrm{pol}} = \mathcal{C}_2 = \mathcal{C}s$.

Example 6.27. Example 6.15 shows that $\mathrm{TI}(\mathcal{S}g^{\triangledown}) = T_3$. Hence $\mathcal{S}g^{\mathrm{pol}} = (\mathcal{S}g^{\triangledown})^{\mathrm{pol}} = \mathcal{C}_3$.

Proposition 6.28. *For any term idempotent variety $\mathcal{V}$ distinct from $\mathcal{A}$, the subvariety $\mathcal{V}^{\mathrm{pol}}$ coincides with the class of $\mathcal{V}$–algebras that have a unique idempotent.*

Proof. Let $\mathcal{K}$ be the class of $\mathcal{V}$–algebras that have a unique idempotent. The pole of any algebra $A \in \mathcal{V}^{\mathrm{pol}}$ is the unique idempotent of A, so $\mathcal{V}^{\mathrm{pol}} \subseteq \mathcal{K}$. Now let $A \in \mathcal{K}$, e be the unique idempotent of A, and $u(\boldsymbol{x}) = v(\boldsymbol{x})$ be an identity of (6.9). Since u and v are term idempotents of $\mathcal{V}$, by Proposition 3.2, for any $\boldsymbol{a} \in A^{\boldsymbol{x}}$, one has $u(\boldsymbol{a}) = v(\boldsymbol{a}) = e$. Hence A satisfies the identity $u = v$. It follows that $A \in \mathcal{V}^{\mathrm{pol}}$, which implies that $\mathcal{K} \subseteq \mathcal{V}^{\mathrm{pol}}$. $\square$

For an algebra A and a sink S of A, let θ_S denote the congruence of A whose only non-singleton congruence class is S. By Proposition 6.1, for a term idempotent variety $\mathcal{V}$ and an algebra $A \in \mathcal{V}$, the set $\mathrm{I}(A)$ of idempotents of A is a sink of A.

Proposition 6.29. *Let $\mathcal{V}$ be a term idempotent variety distinct from $\mathcal{A}$. For any algebra $A \in \mathcal{V}$, the congruence $\theta_{\mathrm{I}(A)}$ of A is the $\mathcal{V}^{\mathrm{pol}}$–replica congruence of A.*

Proof. Let $A \in \mathcal{V}$. By Proposition 6.28, a congruence θ of A is a $\mathcal{V}^{\mathrm{pol}}$–congruence iff A/θ has a unique idempotent. Recall that the element a/θ of A/θ is an idempotent of A/θ iff the congruence class a/θ is a subalgebra of A. By Corollary 3.5, the latter is equivalent to a/θ containing an idempotent of A. It follows that a congruence θ of A is a $\mathcal{V}^{\mathrm{pol}}$–congruence iff one of its congruence classes contains $\mathrm{I}(A)$ as a subset. Since $\theta_{\mathrm{I}(A)}$ is the smallest such congruence, it is the $\mathcal{V}^{\mathrm{pol}}$–replica congruence of A. $\square$

7 Replica congruences

For a variety $\mathcal{V}$ and an algebra A of the same type, we will construct the $\mathcal{V}$–replica congruence of A. Let us define a binary relation $\delta_A^{\mathcal{V}}$ on A as

$$\delta_A^{\mathcal{V}} \;=\; \big\{(u(\boldsymbol{a}), v(\boldsymbol{a})) \mid \mathcal{V} \models u(\boldsymbol{x}) = v(\boldsymbol{x}),\ \boldsymbol{a} \in A^{\boldsymbol{x}}\big\}. \tag{7.1}$$

Observe that $\delta_A^{\mathcal{V}}$ is reflexive (because $\mathcal{V} \models x = x$) and symmetric.

For a binary relation α on a set X, the *transitive closure* $\operatorname{tr}\alpha$ of α is the smallest transitive binary relation on X that contains α. It is given by

$$\operatorname{tr}\alpha = \{(a,b) \in X^2 \mid \exists n \geq 2\ \exists c_1,\ldots,c_n \in X \quad a = c_1\ \alpha\ c_2\ \alpha\ \cdots\ \alpha\ c_n = b\}.$$

We will show that the $\mathcal{V}$–replica congruence of an algebra A coincides with the transitive closure of the relation $\delta_A^{\mathcal{V}}$.

Lemma 7.1. [18, Lem. 3.1] *If α is a reflexive, symmetric, and operation–preserving binary relation on an algebra A, then* $\operatorname{tr}\alpha$ *is a congruence of A.*

Proof. Since $\operatorname{tr}\alpha$ is an equivalence relation, we only need to show that it preserves operations. Let $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$ be such that $(a_i,b_i) \in \operatorname{tr}\alpha$ for every $1 \leq i \leq n$. Then for each $1 \leq i \leq n$ there are a natural number k_i and elements $c_1^i,\ldots,c_{k_i}^i \in A$ such that one has the sequence of relations

$$a_i = c_1^i\ \alpha\ c_2^i\ \alpha\ \cdots\ \alpha\ c_{k_i}^i = b_i.$$

We can assume that all n of these sequences are of equal length k. Otherwise we could use reflexivity to lengthen the sequences that are shorter by repeating their last element a sufficient number of times. Since α preserves operations,

$$f(a_1,\ldots,a_n) = f(c_1^1,\ldots,c_1^n)\ \alpha\ f(c_2^1,\ldots,c_2^n)\ \alpha\ \cdots\ \alpha\ f(c_k^1,\ldots,c_k^n) = f(b_1,\ldots,b_n).$$

Hence $(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \in \operatorname{tr}\alpha$, so $\operatorname{tr}\alpha$ also preserves operations. □

Theorem 7.2. [18, Prop. 3.2] *Let $\mathcal{V}$ be a variety and A be an algebra. The $\mathcal{V}$–replica congruence of A coincides with the transitive closure of $\delta_A^{\mathcal{V}}$.*

Proof. First we will show that $\operatorname{tr}\delta_A^{\mathcal{V}}$ is a $\mathcal{V}$–congruence. Let f be an n–ary basic operation and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ be such that $(a_i, b_i) \in \delta_A^{\mathcal{V}}$ for all $1 \leq i \leq n$. Then for each $1 \leq i \leq n$, there are an identity $u_i(\boldsymbol{x}_i) = v_i(\boldsymbol{x}_i)$ true in $\mathcal{V}$ and $\boldsymbol{c}_i \in A^{\boldsymbol{x}_i}$ such that $a_i = u_i(\boldsymbol{c}_i)$ and $b_i = v_i(\boldsymbol{c}_i)$. It follows that $\mathcal{V}$ also satisfies the identity

$$f(u_1, \ldots, u_n) = f(v_1, \ldots, v_n). \tag{7.2}$$

If we substitute elements $\boldsymbol{c}_i$, $1 \leq i \leq n$, for variables $\boldsymbol{x}_i$, $1 \leq i \leq n$, then the left-hand side and the right-hand side of (7.2) yield elements $f(a_1, \ldots, a_n)$ and $f(b_1, \ldots, b_n)$ respectively. These elements are thus related by $\delta_A^{\mathcal{V}}$, which implies that $\delta_A^{\mathcal{V}}$ preserves operations. Hence, by Lemma 7.1, $\operatorname{tr}\delta_A^{\mathcal{V}}$ is a congruence.

Let $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ be an identity true in $\mathcal{V}$ and let $a_1, \ldots, a_n \in A$. By the definition of $\delta_A^{\mathcal{V}}$, the elements $u(a_1, \ldots, a_n)$ and $v(a_1, \ldots, a_n)$ are related by $\delta_A^{\mathcal{V}}$, and thus also by $\operatorname{tr}\delta_A^{\mathcal{V}}$. Therefore

$$\begin{aligned} u(a_1/\operatorname{tr}\delta_A^{\mathcal{V}}, \ldots, a_n/\operatorname{tr}\delta_A^{\mathcal{V}}) &= u(a_1, \ldots, a_n)/\operatorname{tr}\delta_A^{\mathcal{V}} \\ &= v(a_1, \ldots, a_n)/\operatorname{tr}\delta_A^{\mathcal{V}} = v(a_1/\operatorname{tr}\delta_A^{\mathcal{V}}, \ldots, a_n/\operatorname{tr}\delta_A^{\mathcal{V}}). \end{aligned}$$

Hence $A/\operatorname{tr}\delta_A^{\mathcal{V}}$ satisfies all identities true in $\mathcal{V}$. It follows that $\operatorname{tr}\delta_A^{\mathcal{V}}$ is a $\mathcal{V}$–congruence.

Now we will show that $\operatorname{tr}\delta_A^{\mathcal{V}}$ is the smallest $\mathcal{V}$–congruence. Let θ be a $\mathcal{V}$–congruence and let $(a, b) \in \delta_A^{\mathcal{V}}$. There are an identity $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ true in $\mathcal{V}$ and $c_1, \ldots, c_n \in A$ such that $a = u(c_1, \ldots, c_n)$ and $b = v(c_1, \ldots, c_n)$. Since $A/\theta \in \mathcal{V}$,

$$u(a_1, \ldots, a_n)/\theta = u(a_1/\theta, \ldots, a_n/\theta) = v(a_1/\theta, \ldots, a_n/\theta) = v(a_1, \ldots, a_n)/\theta.$$

Thus $(a, b) \in \theta$, so $\delta_A^{\mathcal{V}} \subseteq \theta$. Hence $\operatorname{tr}\delta_A^{\mathcal{V}} \subseteq \operatorname{tr}\theta = \theta$. $\square$

We can now show that term idempotent varieties $\mathcal{V}$ are characterized by a certain property of $\mathcal{V}$–replica congruences.

Theorem 7.3. [19, Prop. 3.11] *A variety $\mathcal{V}$ of a type Ω is term idempotent iff for any algebra A of the type Ω, every congruence class of the $\mathcal{V}$–replica congruence of A which is not a subalgebra is a singleton.*

Proof. Assume that $\mathcal{V}$ is a term idempotent variety. Let A be an algebra, and let $a \in A$ be such that the congruence class $a/\varrho_A^{\mathcal{V}}$ has more than one element. Then there exists $b \in A$ distinct from a and such that $(a, b) \in \varrho_A^{\mathcal{V}}$. By Theorem 7.2, $\varrho_A^{\mathcal{V}}$ is the transitive closure of $\delta_A^{\mathcal{V}}$, so there exists an element $c \in A$ distinct from a and such that $(a, c) \in \delta_A^{\mathcal{V}}$. This means that there is an identity $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ true in $\mathcal{V}$ and elements $d_1, \ldots, d_n \in A$ such that $a = u(d_1, \ldots, d_n)$ and $c = v(d_1, \ldots, d_n)$. From the fact that a and c are distinct, it follows that $u = v$ is a nontrivial identity. Hence u is a term idempotent of $\mathcal{V}$. Since

$$a/\varrho_A^{\mathcal{V}} = u(d_1, \ldots, d_n)/\varrho_A^{\mathcal{V}} = u(d_1/\varrho_A^{\mathcal{V}}, \ldots, d_n/\varrho_A^{\mathcal{V}}),$$

by Proposition 3.2, $a/\varrho_A^{\mathcal{V}}$ is an idempotent of $A/\varrho_A^{\mathcal{V}} \in \mathcal{V}$. Therefore, by Lemma 2.13, $a/\varrho_A^{\mathcal{V}}$ is a subalgebra of A.

Assume that in every algebra A, each congruence class of the $\mathcal{V}$–replica congruence of A which is not a subalgebra is a singleton. In particular, this is true for the term algebra $T(\omega)$. Consider the free algebra $F_{\mathcal{V}}(\omega) = T(\omega)/\varrho_{T(\omega)}^{\mathcal{V}}$. Let $u = v$ be a nontrivial identity true in $\mathcal{V}$. Then u and v are distinct terms and they are elements of some congruence class C of $\varrho_{T(\omega)}^{\mathcal{V}}$. By the assumption, C is a subalgebra of $T(\omega)$. Thus, by Lemma 2.13, C is an idempotent of $F_{\mathcal{V}}(\omega)$, and so its elements are term idempotents of $\mathcal{V}$. Hence $u = v$ is a term idempotent identity. It follows that $\mathcal{V}$ is a term idempotent variety. □

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If $\mathcal{W}$ is idempotent, then for any algebra $A \in \mathcal{V} \circ \mathcal{W}$, all congruence classes of the $\mathcal{W}$–replica congruence $\varrho_A^{\mathcal{W}}$ are subalgebras of A, which implies that they are $\mathcal{V}$–algebras. If $\mathcal{W}$ is not idempotent, then there may exist congruence classes of $\varrho_A^{\mathcal{W}}$ which are not subalgebras of A. However, if $\mathcal{W}$ is a term idempotent variety, then by Theorem 7.3, all such congruence classes of $\varrho_A^{\mathcal{W}}$ are singletons. We will see that this property ensures that term idempotent varieties behave well as the second factor of the Maltsev product.

In a polarized variety $\mathcal{V}$ of a type Ω, every algebra has a unique idempotent. Hence for any algebra A of the type Ω, there is a unique congruence class of $\varrho_A^{\mathcal{V}}$ which is a subalgebra of A. Therefore one has the following corollary.

Corollary 7.4. *A polarized variety $\mathcal{V}$ of a type Ω is term idempotent iff for any algebra A of the type Ω, there is a unique congruence class of the $\mathcal{V}$–replica congruence $\varrho_A^{\mathcal{V}}$ which is a subalgebra of A, and all the other congruence classes of $\varrho_A^{\mathcal{V}}$ are singletons.*

An algebra A is called *congruence regular* if for any congruences θ and ψ of A, whenever $a/\theta = a/\psi$ for some $a \in A$, the congruences θ and ψ coincide (see [8, Sec. 81]). Equivalently, an algebra A is congruence regular if for any congruence θ of A, whenever $a/\theta = \{a\}$ for some $a \in A$, the congruence θ coincides with the minimum congruence Δ_A of A. E.g. all groups are congruence regular. For a variety $\mathcal{V}$, let $\mathcal{V}^{\mathrm{id}}$ denote the largest idempotent subvariety of $\mathcal{V}$. It is defined relative to $\mathcal{V}$ by the identities $f(x, \ldots, x) = x$, $f \in \Omega$.

Corollary 7.5. *Let $\mathcal{V}$ be a term idempotent variety. If A is a congruence regular algebra that does not belong to $\mathcal{V}$, then the $\mathcal{V}$–replica $A/\varrho_A^{\mathcal{V}}$ of A belongs to $\mathcal{V}^{\mathrm{id}}$.*

Proof. Assume that $\varrho_A^{\mathcal{V}}$ has a singleton congruence class. Then $\varrho_A^{\mathcal{V}} = \Delta_A$. Consequently $A \in \mathcal{V}$, which is a contradiction. Therefore none of the congruence classes of $\varrho_A^{\mathcal{V}}$ are singletons. Hence, by Theorem 7.3, all congruence classes of $\varrho_A^{\mathcal{V}}$ are subalgebras of A. By Lemma 2.13, it follows that $A/\varrho_A^{\mathcal{V}}$ is an idempotent algebra, so it lies in $\mathcal{V}^{\mathrm{id}}$. □

Recall that $\mathcal{T}$ denotes the trivial variety of a given type.

Corollary 7.6. *If $\mathcal{V}$ is a term idempotent variety, then $\mathcal{T} \circ \mathcal{V} = \mathcal{V}$.*

Proof. Let $A \in \mathcal{T} \circ \mathcal{V}$. Consider a congruence class C of $\varrho_A^{\mathcal{V}}$. If C is a subalgebra of A, then $C \in \mathcal{T}$, so C is a singleton. If C is not a subalgebra of A, then by Theorem 7.3, C is still a singleton. Hence $\varrho_A^{\mathcal{V}} = \Delta_A$, which implies that $A \in \mathcal{V}$. □

Theorem 7.2 suggests that replica congruences may sometimes have a simpler form.

Corollary 7.7. *Let $\mathcal{V}$ be a variety and A be an algebra. The $\mathcal{V}$–replica congruence of A coincides with the relation $\delta_A^{\mathcal{V}}$ iff $\delta_A^{\mathcal{V}}$ is transitive.*

For varieties $\mathcal{V}$ and $\mathcal{U}$, we will say that $\mathcal{U}$ *has simple $\mathcal{V}$–replica congruences* if for every $A \in \mathcal{U}$, the relation $\delta_A^{\mathcal{V}}$ is the $\mathcal{V}$–replica congruence of A.

Proposition 7.8. *Let $\mathcal{V}$ and $\mathcal{U}$ be varieties. If $\mathcal{U}$ has simple $\mathcal{V}$–replica congruences, then for every $A \in \mathcal{U}$ and every congruence θ of A, $\theta \vee \varrho_A^{\mathcal{V}} = \theta \circ \varrho_A^{\mathcal{V}} \circ \theta$.*

Proof. Assume that $\mathcal{U}$ has simple $\mathcal{V}$–replica congruences and let $A \in \mathcal{U}$ and θ be a congruence of A. Then $\varrho_A^{\mathcal{V}} = \delta_A^{\mathcal{V}}$ and $\varrho_{A/\theta}^{\mathcal{V}} = \delta_{A/\theta}^{\mathcal{V}}$. Therefore

$$\begin{aligned}
\varrho_{A/\theta}^{\mathcal{V}} &= \delta_{A/\theta}^{\mathcal{V}} \\
&= \{(u(a_1/\theta, \dots, a_n/\theta), v(a_1/\theta, \dots, a_n/\theta)) \mid \mathcal{V} \models u(\boldsymbol{x}) = v(\boldsymbol{x}),\ \boldsymbol{a} \in A^{\boldsymbol{x}}\} \\
&= \{(u(\boldsymbol{a})/\theta, v(\boldsymbol{a})/\theta) \mid \mathcal{V} \models u(\boldsymbol{x}) = v(\boldsymbol{x}),\ \boldsymbol{a} \in A^{\boldsymbol{x}}\} \\
&= \{(a/\theta, b/\theta) \mid (a, b) \in \delta_A^{\mathcal{V}}\} \\
&= \{(a/\theta, b/\theta) \mid (a, b) \in \varrho_A^{\mathcal{V}}\}.
\end{aligned}$$

Hence $(a/\theta, b/\theta) \in \varrho_{A/\theta}^{\mathcal{V}}$ iff there are $(a', b') \in \varrho_A^{\mathcal{V}}$ such that $(a, a'), (b, b') \in \theta$, or equivalently iff $(a, b) \in \theta \circ \varrho_A^{\mathcal{V}} \circ \theta$. By Lemma 2.16, $\varrho_{A/\theta}^{\mathcal{V}} = (\theta \vee \varrho_A^{\mathcal{V}})/\theta$. Thus

$$(a, b) \in \theta \circ \varrho_A^{\mathcal{V}} \circ \theta \iff (a/\theta, b/\theta) \in \varrho_{A/\theta}^{\mathcal{V}} \iff (a, b) \in \theta \vee \varrho_A^{\mathcal{V}},$$

so $\theta \vee \varrho_A^{\mathcal{V}} = \theta \circ \varrho_A^{\mathcal{V}} \circ \theta$. □

We will show that if $\mathcal{V} \subseteq \mathcal{U}$, then the converse is also true.

Lemma 7.9. *Let $\mathcal{V} \subseteq \mathcal{U}$ be varieties and F be a free $\mathcal{U}$–algebra. Then $\delta_F^{\mathcal{V}}$ is the $\mathcal{V}$–replica congruence of F.*

Proof. Let $F = T(X)/\varrho_{T(X)}^{\mathcal{U}}$. By Theorem 7.2, $\delta_F^{\mathcal{V}} \subseteq \varrho_F^{\mathcal{V}}$. We will show that the converse inclusion also holds. Let $s(x_1, \dots, x_n), t(x_1, \dots, x_n) \in T(X)$ be such that $([s], [t]) \in \varrho_F^{\mathcal{V}}$. By Corollary 2.23, $\mathcal{V} \models s = t$. Since $[s] = s([x_1], \dots, [x_n])$ and $[t] = t([x_1], \dots, [x_n])$, one has $([s], [t]) \in \delta_F^{\mathcal{V}}$. Hence $\varrho_F^{\mathcal{V}} \subseteq \delta_F^{\mathcal{V}}$. □

Proposition 7.10. *Let $\mathcal{V} \subseteq \mathcal{U}$ be varieties. The following conditions are equivalent.*

(i) *$\mathcal{U}$ has simple $\mathcal{V}$–replica congruences.*

(ii) *For every $A \in \mathcal{U}$ and any congruence θ of A, one has $\theta \vee \varrho_A^{\mathcal{V}} = \theta \circ \varrho_A^{\mathcal{V}} \circ \theta$.*

(iii) *For every free $\mathcal{U}$–algebra F and any congruence θ of F, one has $\theta \vee \varrho_F^{\mathcal{V}} = \theta \circ \varrho_F^{\mathcal{V}} \circ \theta$.*

Proof. By Proposition 7.8, (i) implies (ii). Clearly (ii) implies (iii). Assume (iii). Let $A \in \mathcal{U}$. Then A is a homomorphic image of some free $\mathcal{U}$–algebra F, i.e. there is a surjective homomorphism $h\colon F \to A$. Suppose $(a,b) \in \varrho_A^{\mathcal{V}}$. There exist $p, q \in F$ such that $h(p) = a$ and $h(q) = b$. By Theorem 2.7, there exists an isomorphism $\varphi\colon F/\ker h \to A$ such that $\varphi(p/\ker h) = a$ and $\varphi(q/\ker h) = b$. Applying Lemma 2.14 to the inverse isomorphism φ^{-1} yields

$$(p/\ker h, q/\ker h) \in \varrho_{F/\ker h}^{\mathcal{V}}.$$

By Lemma 2.16 and by (iii),

$$\varrho_{F/\ker h}^{\mathcal{V}} = (\ker h \vee \varrho_F^{\mathcal{V}})/\ker h = (\ker h \circ \varrho_F^{\mathcal{V}} \circ \ker h)/\ker h.$$

Hence $(p,q) \in \ker h \circ \varrho_F^{\mathcal{V}} \circ \ker h$. Thus there exist $p', q' \in F$ such that

$$(p,p') \in \ker h, \quad (p',q') \in \varrho_F^{\mathcal{V}}, \quad (q',q) \in \ker h.$$

By Lemma 7.9, $\varrho_F^{\mathcal{V}} = \delta_F^{\mathcal{V}}$, so there are an identity $u(x_1,\dots,x_n) = v(x_1,\dots,x_n)$ true in $\mathcal{V}$ and $r_1,\dots,r_n \in F$ such that $p' = u(r_1,\dots,r_n)$ and $q' = v(r_1,\dots,r_n)$. It follows that

$$a = h(p) = h(p') = h(u(r_1,\dots,r_n)) = u(h(r_1),\dots,h(r_n)),$$

$$b = h(q) = h(q') = h(v(r_1,\dots,r_n)) = v(h(r_1),\dots,h(r_n)).$$

Hence $(a,b) \in \delta_A^{\mathcal{V}}$. We have shown that $\varrho_A^{\mathcal{V}} \subseteq \delta_A^{\mathcal{V}}$. Since the converse inclusion always holds, $\varrho_A^{\mathcal{V}} = \delta_A^{\mathcal{V}}$. □

By Corollary 2.24, if $\mathcal{V} \wedge \mathcal{W} \models u = v$, then there exist terms $u = t_1, t_2, \dots, t_n = v$ such that for every $1 \le i < n$, $\mathcal{V} \models t_i = t_{i+1}$ or $\mathcal{W} \models t_i = t_{i+1}$. The following proposition shows that if $\mathcal{V} \vee \mathcal{W}$ has simple $\mathcal{W}$–replica congruences, then this chain of identities has a particular form.

Proposition 7.11. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If $\mathcal{V} \vee \mathcal{W}$ has simple $\mathcal{W}$–replica congruences, then for every identity $u(\boldsymbol{x}) = v(\boldsymbol{x})$ true in $\mathcal{V} \wedge \mathcal{W}$, there exist terms $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ such that*

$$\mathcal{V} \models u = p, \quad \mathcal{W} \models p = q, \quad \mathcal{V} \models q = v.$$

Proof. Let F be the free algebra $T(\boldsymbol{x})/\varrho_{T(\boldsymbol{x})}^{\mathcal{V}\vee\mathcal{W}}$ of $\mathcal{V}\vee\mathcal{W}$ over the set $\boldsymbol{x}$. Let $u(\boldsymbol{x}) = v(\boldsymbol{x})$ be an identity true in $\mathcal{V}\wedge\mathcal{W}$. By Corollary 2.23, $([u],[v]) \in \varrho_F^{\mathcal{V}\wedge\mathcal{W}}$. Furthermore, by Lemma 2.15, $\varrho_F^{\mathcal{V}\wedge\mathcal{W}} = \varrho_F^{\mathcal{V}} \vee \varrho_F^{\mathcal{W}}$, and by Proposition 7.10, $\varrho_F^{\mathcal{V}} \vee \varrho_F^{\mathcal{W}} = \varrho_F^{\mathcal{V}} \circ \varrho_F^{\mathcal{W}} \circ \varrho_F^{\mathcal{V}}$. Hence there exist terms $p, q \in T(\boldsymbol{x})$ such that $[u]\ \varrho_F^{\mathcal{V}}\ [p]\ \varrho_F^{\mathcal{W}}\ [q]\ \varrho_F^{\mathcal{V}}\ [v]$. The conclusion follows by Corollary 2.23. $\square$

Since $\mathcal{V}\vee\mathcal{W} \subseteq \mathsf{H}(\mathcal{V}\circ\mathcal{W})$, Proposition 7.11 provides a necessary condition for $\mathsf{H}(\mathcal{V}\circ\mathcal{W})$ to have simple $\mathcal{W}$–replica congruences. On the other hand, the following proposition provides a sufficient condition for $\mathsf{H}(\mathcal{V}\circ\mathcal{W})$ to have simple $\mathcal{W}$–replica congruences.

Proposition 7.12. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ be term idempotent. If there exist terms $p(x,y,z)$, $q(x,y,z)$, and $t(x)$ such that the following conditions are satisfied*

(a) $\mathcal{V} \models p(x,y,y) = x,\ q(x,x,y) = y,$

(b) $\mathcal{W} \models p(t(x),t(x),t(y)) = q(t(x),t(y),t(y)),$

then $\mathsf{H}(\mathcal{V}\circ\mathcal{W})$ has simple $\mathcal{W}$–replica congruences.

Proof. Let $A \in \mathsf{H}(\mathcal{V}\circ\mathcal{W})$. We will show that $\delta_A^{\mathcal{W}}$ is transitive. Let $(a,b) \in \delta_A^{\mathcal{W}}$ and $(b,c) \in \delta_A^{\mathcal{W}}$. If b coincides with a or c, then $(a,c) \in \delta_A^{\mathcal{W}}$, so we can assume that b is distinct from a and c. It follows that there are nontrivial identities $u_1(\boldsymbol{x}_1) = v_1(\boldsymbol{x}_1)$ and $u_2(\boldsymbol{x}_2) = v_2(\boldsymbol{x}_2)$ true in $\mathcal{W}$ and elements $\boldsymbol{d}_1 \in A^{\boldsymbol{x}_1}$ and $\boldsymbol{d}_2 \in A^{\boldsymbol{x}_2}$ such that

$$a = u_1(\boldsymbol{d}_1), \quad b = v_1(\boldsymbol{d}_1),$$

$$b = u_2(\boldsymbol{d}_2), \quad c = v_2(\boldsymbol{d}_2).$$

The terms u_1, v_1, u_2, and v_2 are term idempotents of $\mathcal{W}$, so $\mathcal{W}$ satisfies the identities $t(u_1) = u_1$, $t(v_1) = v_1$, $t(u_2) = u_2$, and $t(v_2) = v_2$. We define the terms

$$u(\boldsymbol{x}_1,\boldsymbol{x}_2) = p(u_1,v_1,u_2) \qquad \text{and} \qquad v(\boldsymbol{x}_1,\boldsymbol{x}_2) = q(v_1,u_2,v_2).$$

The following identities are true in $\mathcal{W}$,

$$\begin{aligned}\mathcal{W} \models u = p(u_1,v_1,u_2) &= p(u_1,u_1,u_2) = p(t(u_1),t(u_1),t(u_2))\\ &= q(t(u_1),t(u_2),t(u_2)) = q(u_1,u_2,u_2) = q(v_1,u_2,v_2) = v,\end{aligned}$$

where the middle identity follows from (b). Hence $\mathcal{W}$ satisfies the identity $u = v$.

By Lemma 4.1, (a) implies that A satisfies the identities

$$p(u_1, v_1, v_1) = u_1, \quad q(u_2, u_2, v_2) = v_2.$$

Consequently,

$$u(\boldsymbol{d}_1, \boldsymbol{d}_2) = p(u_1(\boldsymbol{d}_1), v_1(\boldsymbol{d}_1), u_2(\boldsymbol{d}_2)) = p(u_1(\boldsymbol{d}_1), v_1(\boldsymbol{d}_1), v_1(\boldsymbol{d}_1)) = u_1(\boldsymbol{d}_1) = a,$$

$$v(\boldsymbol{d}_1, \boldsymbol{d}_2) = q(v_1(\boldsymbol{d}_1), u_2(\boldsymbol{d}_2), v_2(\boldsymbol{d}_2)) = q(u_2(\boldsymbol{d}_2), u_2(\boldsymbol{d}_2), v_2(\boldsymbol{d}_2)) = v_2(\boldsymbol{d}_2) = c.$$

We have shown that $(a, c) \in \delta_A^{\mathcal{W}}$, so $\delta_A^{\mathcal{W}}$ is transitive. By Corollary 7.7, $\varrho_A^{\mathcal{W}} = \delta_A^{\mathcal{W}}$. □

For varieties $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, the implication of Proposition 7.12 becomes an equivalence and the conditions (a) and (b) reduce to a simpler form.

Proposition 7.13. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, and let $\mathcal{W}$ be term idempotent. The variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has simple $\mathcal{W}$–replica congruences iff there exist terms $p(x, y)$ and $q(x, y)$ such that the following conditions are satisfied*

(a) $\mathcal{V} \models p(x, y) = x,\ q(x, y) = y$,

(b) $\mathcal{W} \models p(x, y) = q(x, y)$.

Proof. The conditions (a) and (b) are a special case of the conditions (a) and (b) of Proposition 7.12. Namely the case when the term $t(x)$ is the variable x and the terms $p(x, y, z)$ and $q(x, y, z)$ do not depend on the middle variable. Therefore (a) and (b) imply that $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has simple $\mathcal{W}$–replica congruences.

Assume that $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has simple $\mathcal{W}$–replica congruences. Then $\mathcal{V} \vee \mathcal{W}$ also has simple $\mathcal{W}$–replica congruences, because $\mathcal{V} \vee \mathcal{W} \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$. Since $\mathcal{V} \wedge \mathcal{W}$ is trivial, it satisfies the identity $x = y$. By Proposition 7.11, there are terms $p(x, y)$ and $q(x, y)$ such that (a) and (b) are satisfied. □

For a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, there is a description of the $\mathcal{W}$–replica congruence of an algebra $A \in \mathcal{V} \circ \mathcal{W}$ which is different from the one provided by Theorem 7.2. The set $\mathrm{Id}(\mathcal{V}) \cup \mathrm{Id}(\mathcal{W})$ is an equational base for the trivial variety $\mathcal{V} \wedge \mathcal{W}$. Hence one of its consequences is the identity $x = y$. Furthermore, the identity $x = y$ contains only the variables x and y, so it is a consequence of $\mathrm{Id}_{\{x,y\}}(\mathcal{V}) \cup \mathrm{Id}_{\{x,y\}}(\mathcal{W})$, where $\mathrm{Id}_{\{x,y\}}(\mathcal{U})$ is the subset of $\mathrm{Id}(\mathcal{U})$ consisting of identities whose both sides belong to $T(\{x, y\})$.

For a set X, let $\Delta_X = \{(a, a) \mid a \in X\}$. For any binary relation $\alpha \subseteq X^2$, the *reflexive closure* of α is the smallest reflexive relation that contains α. It is given by $\alpha \cup \Delta_X$.

Theorem 7.14. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, and let $\mathcal{W}$ be term idempotent. Let $\Sigma \subseteq \mathrm{Id}_{\{x,y\}}(\mathcal{V})$ be such that the identity $x = y$ is a consequence of $\Sigma \cup \mathrm{Id}_{\{x,y\}}(\mathcal{W})$. Then for every algebra $A \in \mathcal{V} \circ \mathcal{W}$, the $\mathcal{W}$–replica congruence of A is given by*

$$\varrho_A^{\mathcal{W}} \;=\; \{(a, b) \in A^2 \mid \; \forall (u(x, y) = v(x, y)) \in \Sigma \;\; u(a, b) = v(a, b)\} \cup \Delta_A.$$

Proof. Let $A \in \mathcal{V} \circ \mathcal{W}$ and let ψ be the binary relation from the statement of the theorem. By Theorem 7.3, every congruence class of $\varrho_A^{\mathcal{W}}$ is a subalgebra of A, and thus a $\mathcal{V}$–algebra, or it is a singleton. Hence every pair of distinct elements $(a, b) \in \varrho_A^{\mathcal{W}}$ satisfies every identity in Σ, which implies that $\varrho_A^{\mathcal{W}} \subseteq \psi$. Now let $(a, b) \in \psi$ and $a \neq b$. Then a and b satisfy every identity in Σ. Consequently, the elements $a/\varrho_A^{\mathcal{W}}$ and $b/\varrho_A^{\mathcal{W}}$ of $A/\varrho_A^{\mathcal{W}}$ also satisfy every identity in Σ. Furthermore, since $A/\varrho_A^{\mathcal{W}} \in \mathcal{W}$, the elements $a/\varrho_A^{\mathcal{W}}$ and $b/\varrho_A^{\mathcal{W}}$ satisfy every identity in $\mathrm{Id}_{\{x,y\}}(\mathcal{W})$. It follows that $a/\varrho_A^{\mathcal{W}} = b/\varrho_A^{\mathcal{W}}$. Hence $(a, b) \in \varrho_A^{\mathcal{W}}$, so $\psi \subseteq \varrho_A^{\mathcal{W}}$. $\square$

If $\mathcal{W}$ is idempotent, then for every $A \in \mathcal{V} \circ \mathcal{W}$ and $a \in A$, the congruence class $a/\varrho_A^{\mathcal{W}}$ is a subalgebra of A, and so $a/\varrho_A^{\mathcal{W}} \in \mathcal{V}$. Hence for any identity $u(x, y) = v(x, y)$ true in $\mathcal{V}$, one has $u(a, a) = v(a, a)$. Thus for idempotent $\mathcal{W}$, the reflexive closure in the formula for $\varrho_A^{\mathcal{W}}$ in Theorem 7.14 is redundant and it may be dropped.

Example 7.15. Let $\mathcal{V}$ be a variety which satisfies an irregular identity $t(x, y) = u(x)$, where the term $t(x, y)$ is binary. Then the intersection of $\mathcal{V}$ with the variety $\mathcal{S}$ of Ω–semilattices is trivial. The set of identities $\{t(x, y) = u(x)\} \cup \mathrm{Id}_{\{x,y\}}(\mathcal{S})$ implies the identity $x = y$, because one has

$$x = u(x) = t(x, y) = t(y, x) = u(y) = y.$$

Hence, by Theorem 7.14, for every $A \in \mathcal{V} \circ \mathcal{S}$, the $\mathcal{S}$–replica congruence of A is given by

$$\varrho_A^{\mathcal{S}} \;=\; \{(a,b) \in A^2 \mid t(a,b) = u(a)\}.$$

8 Sufficient condition

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, $\mathcal{F}$ be the class of free algebras of $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$, $\mathcal{K}$ be a class such that $\mathcal{F} \subseteq \mathcal{K} \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$, and n be a positive integer. We will denote by $P_{\mathcal{K}}(n)$ the condition that for any algebra $A \in \mathcal{K}$, any congruence θ of A, any congruence class D of $\theta \vee \varrho_A^{\mathcal{W}}$ which is a subalgebra of A, and any elements $a_1, \ldots, a_n \in D$, there exist a congruence class E of $\varrho_A^{\mathcal{W}}$ which is a subalgebra of A and elements $a_1', \ldots, a_n' \in E$ such that $(a_i, a_i') \in \theta$ for each $1 \leq i \leq n$. By Theorem 2.18, $\mathcal{F} \subseteq \mathcal{V} \circ \mathcal{W} \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$. We will denote the condition $P_{\mathcal{V} \circ \mathcal{W}}(n)$ by $P(n)$.

The following proposition provides a method of proving that a given condition on varieties $\mathcal{V}$ and $\mathcal{W}$ is sufficient for the Maltsev product $\mathcal{V} \circ \mathcal{W}$ to be a variety. It suffices to prove that this condition implies the conditions $P(n)$, $n \geq 1$.

Proposition 8.1. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If $P(n)$ holds for every $n \geq 1$, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. In order to prove that the prevariety $\mathcal{V} \circ \mathcal{W}$ is closed under homomorphic images, we will show that each quotient of an algebra $A \in \mathcal{V} \circ \mathcal{W}$ belongs to $\mathcal{V} \circ \mathcal{W}$. Let θ be a congruence of A and C be a congruence class of $\varrho_{A/\theta}^{\mathcal{W}}$ which is a subalgebra of A/θ. Let an identity $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ be true in $\mathcal{V}$ and let $a_1/\theta, \ldots, a_n/\theta \in C$. By Lemma 2.16, one has $\varrho_{A/\theta}^{\mathcal{W}} = (\theta \vee \varrho_A^{\mathcal{W}})/\theta$. By Lemma 2.10, the congruence class $\bigcup C$ of $\theta \vee \varrho_A^{\mathcal{W}}$ is a subalgebra of A. By $P(n)$, there are a congruence class E of $\varrho_A^{\mathcal{W}}$ which is a subalgebra of A and elements $a_1', \ldots, a_n' \in E$ such that $(a_i, a_i') \in \theta$ for each $1 \leq i \leq n$. By Theorem 2.25, $E \in \mathcal{V}$, which implies that $u(a_1', \ldots, a_n') = v(a_1', \ldots, a_n')$. Therefore

$$\begin{aligned} u(a_1/\theta, \ldots, a_n/\theta) &= u(a_1'/\theta, \ldots, a_n'/\theta) = u(a_1', \ldots, a_n')/\theta \\ &= v(a_1', \ldots, a_n')/\theta = v(a_1'/\theta, \ldots, a_n'/\theta) = v(a_1/\theta, \ldots, a_n/\theta). \end{aligned}$$

Hence C satisfies every identity true in $\mathcal{V}$, and thus $C \in \mathcal{V}$. It follows that $A/\theta \in \mathcal{V} \circ \mathcal{W}$. □

In the proof of Proposition 8.1 we make use of the fact that every algebra in $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is a homomorphic image of some algebra in $\mathcal{V} \circ \mathcal{W}$. Due to this fact, the assumption that the conditions $P_{\mathcal{V} \circ \mathcal{W}}(n)$, $n \geq 1$, hold, allows us derive the conclusion. Every algebra in $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is also a homomorphic image of some algebra in $\mathcal{F} \subseteq \mathcal{V} \circ \mathcal{W}$. Thus we could derive the same conclusion assuming only that the conditions $P_{\mathcal{F}}(n)$, $n \geq 1$, hold. However the following result shows that the conditions $P_{\mathcal{F}}(n)$ and $P_{\mathcal{V}\circ\mathcal{W}}(n)$ are equivalent for every $n \geq 1$, so the latter approach is not more general.

Proposition 8.2. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and let $\mathcal{F} \subseteq \mathcal{K}, \mathcal{K}' \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$. For every positive integer n, the conditions $P_{\mathcal{K}}(n)$ and $P_{\mathcal{K}'}(n)$ are equivalent.*

Proof. We will show that $P_{\mathcal{F}}(n)$ implies $P_{\mathsf{H}(\mathcal{V}\circ\mathcal{W})}(n)$. Assume $P_{\mathcal{F}}(n)$. Each algebra in $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ is isomorphic to some quotient F/ψ of some free algebra $F \in \mathcal{F}$. By Theorem 2.9, every congruence of F/ψ is of the form θ/ψ for some congruence $\theta \supseteq \psi$ of F. The assignment $\theta \mapsto \theta/\psi$ is a lattice isomorphism, so it preserves joins. Let D be a congruence class of $\theta/\psi \vee \varrho^{\mathcal{W}}_{F/\psi}$ which is a subalgebra of F/ψ and $a_1/\psi, \dots, a_n/\psi \in D$. By Lemma 2.16, $\varrho^{\mathcal{W}}_{F/\psi} = (\psi \vee \varrho^{\mathcal{W}}_F)/\psi$. Thus

$$\theta/\psi \vee \varrho^{\mathcal{W}}_{F/\psi} = \theta/\psi \vee (\psi \vee \varrho^{\mathcal{W}}_F)/\psi = (\theta \vee \psi \vee \varrho^{\mathcal{W}}_F)/\psi = (\theta \vee \varrho^{\mathcal{W}}_F)/\psi.$$

By Lemma 2.10, the congruence class $\bigcup D$ of $\theta \vee \varrho^{\mathcal{W}}_F$ is a subalgebra of F. By $P_{\mathcal{F}}(n)$, there exist $a'_1, \dots, a'_n \in \bigcup D$ that lie in the same congruence class of $\varrho^{\mathcal{W}}_F$ which is a subalgebra of F, and are such that $(a_i, a'_i) \in \theta$ for each $1 \leq i \leq n$. Consequently $a'_1/\psi, \dots, a'_n/\psi \in D$ lie in the same congruence class of $(\psi \vee \varrho^{\mathcal{W}}_F)/\psi = \varrho^{\mathcal{W}}_{F/\psi}$ and $(a_i/\psi, a'_i/\psi) \in \theta/\psi$ for each $1 \leq i \leq n$. It follows that $P_{\mathsf{H}(\mathcal{V}\circ\mathcal{W})}(n)$ holds.

Now let $\mathcal{F} \subseteq \mathcal{K}, \mathcal{K}' \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$. Then $P_{\mathcal{K}}(n)$ implies $P_{\mathcal{F}}(n)$, $P_{\mathcal{F}}(n)$ implies $P_{\mathsf{H}(\mathcal{V}\circ\mathcal{W})}(n)$, and $P_{\mathsf{H}(\mathcal{V}\circ\mathcal{W})}(n)$ implies $P_{\mathcal{K}'}(n)$. Hence $P_{\mathcal{K}}(n)$ implies $P_{\mathcal{K}'}(n)$. By the same argument $P_{\mathcal{K}'}(n)$ implies $P_{\mathcal{K}}(n)$. $\square$

The proofs of the following two results show the usefulness of the notion of term idempotence in the context of this chapter. Recall that the variety $\mathcal{A}$ of all algebras of a given type is the only term idempotent variety that has no term idempotents.

Proposition 8.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. If $\mathcal{W}$ is term idempotent and distinct from $\mathcal{A}$, then the condition $P(1)$ holds.*

Proof. Let $A \in \mathcal{V} \circ \mathcal{W}$, θ be a congruence of A, D be a congruence class of $\theta \vee \varrho_A^{\mathcal{W}}$ which is a subalgebra of A, and $a \in D$. Assume that there is no $a' \in a/\theta$ such that the congruence class $a'/\varrho_A^{\mathcal{W}}$ is a subalgebra of A. By Theorem 7.3, for every $b \in a/\theta$, one has $b/\varrho_A^{\mathcal{W}} = \{b\} \subseteq a/\theta$. Hence $a/\theta = a/(\theta \vee \varrho_A^{\mathcal{W}}) = D$.

Let $t(x)$ be a term idempotent of $\mathcal{W}$. Since $D = a/\theta$ is a subalgebra of A, it contains the value $t(a)$. By Lemma 3.3, the congruence class $t(a)/\varrho_A^{\mathcal{W}}$ is a subalgebra of A, which is a contradiction. It follows that $P(1)$ holds. □

Proposition 8.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ be term idempotent and distinct from $\mathcal{A}$. The condition $P(2)$ holds iff $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has simple $\mathcal{W}$–replica congruences.*

Proof. We apply Proposition 7.10 to varieties $\mathcal{W} \subseteq \mathsf{H}(\mathcal{V} \circ \mathcal{W})$. The condition (i), which says that $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has simple $\mathcal{W}$–replica congruences, is equivalent to the condition (ii), which says that for every $A \in \mathsf{H}(\mathcal{V} \circ \mathcal{W})$ and any congruence θ of A, one has $\theta \vee \varrho_A^{\mathcal{W}} = \theta \circ \varrho_A^{\mathcal{W}} \circ \theta$.

Assume that $P(2)$ holds. By Proposition 8.2, $P_{\mathsf{H}(\mathcal{V} \circ \mathcal{W})}(2)$ also holds. Let $A \in \mathsf{H}(\mathcal{V} \circ \mathcal{W})$ and θ be a congruence of A. By Theorem 7.3, congruence classes of $\varrho_{A/\theta}^{\mathcal{W}} = (\theta \vee \varrho_A^{\mathcal{W}})/\theta$ are subalgebras of A/θ or singletons. Thus, by Lemma 2.10, congruence classes of $\theta \vee \varrho_A^{\mathcal{W}}$ are subalgebras of A or congruence classes of θ. Let $(a, b) \in \theta \vee \varrho_A^{\mathcal{W}}$. Then a and b lie in some congruence class D of $\theta \vee \varrho_A^{\mathcal{W}}$. If D is a subalgebra, then by $P_{\mathsf{H}(\mathcal{V} \circ \mathcal{W})}(2)$, there are $a', b' \in D$ such that $(a', b') \in \varrho_A^{\mathcal{W}}$ and $(a, a'), (b, b') \in \theta$. Hence $(a, b) \in \theta \circ \varrho_A^{\mathcal{W}} \circ \theta$. Otherwise, D is a congruence class of θ, so $(a, b) \in \theta \subseteq \theta \circ \varrho_A^{\mathcal{W}} \circ \theta$. It follows that $\theta \vee \varrho_A^{\mathcal{W}} = \theta \circ \varrho_A^{\mathcal{W}} \circ \theta$.

Assume that for each $A \in \mathsf{H}(\mathcal{V} \circ \mathcal{W})$ and any congruence θ of A, $\theta \vee \varrho_A^{\mathcal{W}} = \theta \circ \varrho_A^{\mathcal{W}} \circ \theta$. Let $A \in \mathcal{V} \circ \mathcal{W}$, θ be a congruence of A, D be a congruence class of $\theta \vee \varrho_A^{\mathcal{W}}$ which is a subalgebra of A, and $a, b \in D$. Then $(a, b) \in \theta \circ \varrho_A^{\mathcal{W}} \circ \theta$, which implies that there are $c, d \in D$ such that $a \mathrel{\theta} c \mathrel{\varrho_A^{\mathcal{W}}} d \mathrel{\theta} b$. By Theorem 7.3, the congruence class $c/\varrho_A^{\mathcal{W}} = d/\varrho_A^{\mathcal{W}}$ is a subalgebra of A or a singleton. In the former case $P(2)$ holds with $a' = c$ and $b' = d$. In the latter case $c = d$, so $(a, b) \in \theta$. By Proposition 8.3, $P(1)$ holds. Thus there exists $e \in D$ such that $(a, e) \in \theta$ and $e/\varrho_A^{\mathcal{W}}$ is a subalgebra. Since also $(b, e) \in \theta$, $P(2)$ holds with $a' = b' = e$. □

We will now use the method provided by Proposition 8.1 to prove a new sufficient condition for the Maltsev product $\mathcal{V} \circ \mathcal{W}$ of a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ to be a variety. Note that by Proposition 8.4, any such sufficient condition which is provable using that method must imply that $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has simple $\mathcal{W}$–replica congruences.

Lemma 8.5. *Let $\mathcal{W}$ be a variety, u and v be terms, and $t(x)$ be an at most unary term. If*

$$\mathcal{W} \models u(t(x_1), \ldots, t(x_n)) = v(t(x_1), \ldots, t(x_n)), \tag{8.1}$$

then for every at most unary term idempotent $p(x)$ of $\mathcal{W}$,

$$\mathcal{W} \models u(p(x_1), \ldots, p(x_n)) = v(p(x_1), \ldots, p(x_n)). \tag{8.2}$$

Proof. Let $p(x)$ be a term idempotent of $\mathcal{W}$. The identity (8.1) has the consequence

$$\mathcal{W} \models u(t(p(x_1)), \ldots, t(p(x_n))) = v(t(p(x_1)), \ldots, t(p(x_n))). \tag{8.3}$$

If t is unary, then by Proposition 3.7(3), $t(p(x))$ is $\mathcal{W}$–equivalent to $p(x)$. If t is nullary, then $t(p(x))$ coincides with the constant term t. Thus, by Proposition 3.11, $t(p(x))$ is $\mathcal{W}$–equivalent to $p(x)$. Therefore (8.3) implies (8.2). □

Theorem 8.6. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ be term idempotent. If there exist terms $p(x, y, z)$, $q(x, y, z)$, and $t(x)$ such that*

(a) $\mathcal{V} \models p(x, y, y) = x,\ q(x, x, y) = y,$

(b) $\mathcal{W} \models p(t(x), t(x), t(y)) = q(t(x), t(y), t(y)),$

then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Proof. For any variety $\mathcal{V}$, the Maltsev product $\mathcal{V} \circ \mathcal{A}$ coincides with $\mathcal{A}$, so it is a variety. We may thus assume that $\mathcal{W}$ is distinct from $\mathcal{A}$. By Lemma 8.5, we may assume that $t(x)$ is a term idempotent of $\mathcal{W}$.

By Proposition 8.3, $P(1)$ holds. By Proposition 7.12 and Proposition 8.4, $P(2)$ holds. Let $n \geq 3$ and assume that $P(n-1)$ holds. We will show that $P(n)$ also holds. Let $A \in \mathcal{V} \circ \mathcal{W}$, θ be a congruence of A, D be a congruence class of $\theta \vee \varrho_A^{\mathcal{W}}$ which is a subalgebra of A, and

$a_1, \ldots, a_n \in D$. By $P(n-1)$, there are a congruence class E of $\varrho_A^{\mathcal{W}}$ which is a subalgebra of A and elements $b_1, \ldots, b_{n-1} \in E$ such that $(a_i, b_i) \in \theta$ for each $1 \leq i < n$. By $P(2)$, there are a congruence class E' of $\varrho_A^{\mathcal{W}}$ which is a subalgebra of A and elements $c, d \in E'$ such that $(b_1, c) \in \theta$ and $(a_n, d) \in \theta$. By Theorem 2.25, E and E' belong to $\mathcal{V}$, so they satisfy the identities of (a). Furthermore, $A/\varrho_A^{\mathcal{W}}$ satisfies the identity of (b). It follows that for every $1 \leq i < n$,

$$\begin{aligned} a_i \,\theta\, b_i = p(b_i, b_1, b_1) \,\theta\, p(b_i, b_1, c) \,\varrho_A^{\mathcal{W}}\, p(b_1, b_1, d) \,\varrho_A^{\mathcal{W}}\, p(t(b_1), t(b_1), t(d)) \\ \varrho_A^{\mathcal{W}}\, q(t(b_1), t(d), t(d)) \,\varrho_A^{\mathcal{W}}\, q(b_1, d, d) \,\varrho_A^{\mathcal{W}}\, q(b_1, c, d) \,\theta\, q(c, c, d) = d \,\theta\, a_n. \end{aligned}$$

Hence if we define $a_i' = p(b_i, b_1, c)$ for each $1 \leq i < n$ and $a_n' = q(b_1, c, d)$, then

$$a_i \,\theta\, a_i' \,\varrho_A^{\mathcal{W}}\, p(t(b_1), t(b_1), t(d)), \qquad \forall 1 \leq i \leq n.$$

Therefore elements $a_1', \ldots, a_n'$ lie in the congruence class $p(t(b_1), t(b_1), t(d))/\varrho_A^{\mathcal{W}}$ and are such that $(a_i, a_i') \in \theta$ for each $1 \leq i \leq n$. By Proposition 6.1, the set of term idempotents of $\mathcal{W}$ is a sink of $T(\omega)$. Since the term $t(x)$ is a term idempotent of $\mathcal{W}$, the term $p(t(x), t(x), t(y))$ is also a term idempotent of $\mathcal{W}$. Thus by Proposition 3.3, the congruence class $p(t(b_1), t(b_1), t(d))/\varrho_A^{\mathcal{W}}$ is a subalgebra of A. Consequently $P(n)$ holds. By Proposition 8.1, $\mathcal{V} \circ \mathcal{W}$ is a variety. □

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties that satisfy the sufficient condition presented in Theorem 8.6. By Theorem 4.2, the variety $\mathcal{V} \circ \mathcal{W}$ is defined by the set of identities $\Sigma^{\mathcal{W}}$ for any equational base Σ for $\mathcal{V}$. By Theorem 7.3, the $\mathcal{W}$–replica congruence $\varrho_A^{\mathcal{W}}$ of an algebra $A \in \mathcal{V} \circ \mathcal{W}$ partitions the universe of A into congruence classes that are $\mathcal{V}$–algebras or singletons. By Proposition 7.12,

$$\varrho_A^{\mathcal{W}} = \big\{(u(\boldsymbol{a}),\ v(\boldsymbol{a})) \mid \mathcal{W} \models u(\boldsymbol{x}) = v(\boldsymbol{x}),\ \boldsymbol{a} \in A^{\boldsymbol{x}}\big\}.$$

For any variety $\mathcal{U}$ that contains both $\mathcal{V}$ and $\mathcal{W}$, the Maltsev $\mathcal{U}$–product $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$ coincides with the intersection of the varieties $\mathcal{V} \circ \mathcal{W}$ and $\mathcal{U}$. Hence $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$ is a variety. It is defined relative to $\mathcal{U}$ by $\Sigma^{\mathcal{W}}$. For any subvarieties $\mathcal{V}_0 \subseteq \mathcal{V}$ and $\mathcal{W}_0 \subseteq \mathcal{W}$, the varieties $\mathcal{V}_0$ and $\mathcal{W}_0^{\triangledown}$ also satisfy the sufficient condition, so the Maltsev product $\mathcal{V}_0 \circ \mathcal{W}_0^{\triangledown}$ is a variety.

Theorem 8.6 is a considerable extension of [19, Thm. 4.1]. The new proof strategy employed in this work allowed us to obtain a weaker sufficient condition. We will finish this chapter with comments on some aspects of the new sufficient condition.

The use of the term $t(x)$ in the condition (b) may be interpreted as a requirement that for any algebra $A \in \mathcal{W}$, the identity

$$p(x, x, y) = q(x, y, y) \tag{8.4}$$

is satisfied by the elements of the set $S = \{t(a) \mid a \in A\}$. The set S contains all idempotents of A. If $t(x)$ is a term idempotent of $\mathcal{W}$, then S is precisely the set $\mathrm{I}(A)$ of idempotents of A. By Proposition 6.1, $\mathrm{I}(A)$ is a subalgebra of A. We thus only require that the subalgebras $\mathrm{I}(A)$ of algebras $A \in \mathcal{W}$ satisfy the identity (8.4).

The conditions (a) and (b) of Theorem 8.6 resemble the Maltsev condition for a variety to be congruence 3–permutable. By Theorem 2.12, a variety $\mathcal{U}$ is congruence 3–permutable iff there exist terms $p(x, y, z)$ and $q(x, y, z)$ such that $\mathcal{U}$ satisfies the identities

$$p(x, y, y) = x, \quad q(x, x, y) = y, \quad p(x, x, y) = q(x, y, y). \tag{8.5}$$

The condition (a) requires $\mathcal{V}$ to satisfy the first two of identities (8.5) and the condition (b) requires $\mathcal{W}$ to satisfy a weaker version of the last of identities (8.5). This similarity may be explained by the following observation. The identities (8.5) entail that for any congruences θ and ψ of an algebra $A \in \mathcal{U}$, one has $\theta \vee \psi = \theta \circ \psi \circ \theta$. On the other hand by Propositions 7.10 and 7.12, the conditions (a) and (b) entail that for any congruence θ of an algebra $A \in \mathcal{V} \circ \mathcal{W}$, one has $\theta \vee \varrho_A^{\mathcal{W}} = \theta \circ \varrho_A^{\mathcal{W}} \circ \theta$.

9 Consequences and examples

We will derive additional sufficient conditions as consequences of Theorem 8.6 and we will illustrate their application with examples. In the following two results, the requirements for varieties $\mathcal{V}$ and $\mathcal{W}$ are separated, so these varieties may be chosen independently.

Theorem 9.1. *If $\mathcal{V}$ is a congruence permutable variety and $\mathcal{W}$ is a term idempotent variety, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. Let $p(x, y, z)$ be a Maltsev term for $\mathcal{V}$. Define terms $q(x, y, z)$ and $t(x)$ as $p(x, x, z)$ and x respectively. For such terms p, q, and t, the conditions of Theorem 8.6 take the following form:

(a) $\mathcal{V} \models p(x, y, y) = x,\ p(x, x, y) = y,$

(b) $\mathcal{W} \models p(x, x, y) = p(x, x, y).$

The identities of (a) are the identities that define a Maltsev term, so (a) is satisfied. The identity of (b) is trivial, so (b) is satisfied. □

Example 9.2. [19, Ex. 5.8] The variety $\mathcal{G}$ of groups of the type $\{\,\cdot\,,^{-1}\,\}$ is defined by the following identities:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z),$

(2) $x \cdot x^{-1} = y \cdot y^{-1},$

(3) $x \cdot (x \cdot x^{-1}) = x = (x \cdot x^{-1}) \cdot x.$

This variety is equivalent to the usual variety of groups of the type $\{\,\cdot\,,^{-1}\,, e\}$. We will consider the Maltsev product of $\mathcal{G}$ and the variety of lattices $\mathcal{L}$. We must first present each of them as an equivalent variety of the type $\Omega = \{+,\ \cdot\,,^{-1}\,\}$. The variety $\overline{\mathcal{G}}$ of groups of the type Ω is defined by the identities that define $\mathcal{G}$ and the identity $x + y = x \cdot y$. The variety $\overline{\mathcal{L}}$ of lattices of the type Ω is defined by the identities that define $\mathcal{L}$ and the identity $x^{-1} = x$. Since $\overline{\mathcal{G}}$ is congruence permutable and $\overline{\mathcal{L}}$ is idempotent, by Theorem 9.1, the Maltsev product $\overline{\mathcal{G}} \circ \overline{\mathcal{L}}$ is a variety. Both $\overline{\mathcal{G}}$ and $\overline{\mathcal{L}}$ are strongly irregular and $\overline{\mathcal{L}}$ is idempotent, so by Proposition 4.15, $\overline{\mathcal{G}} \circ \overline{\mathcal{L}}$ is a strongly irregular variety. Furthermore, for every pair of subvarieties $\mathcal{G}' \subseteq \overline{\mathcal{G}}$ and $\mathcal{L}' \subseteq \overline{\mathcal{L}}$, their Maltsev product $\mathcal{G}' \circ \mathcal{L}'$ is also a variety. E.g. $\mathcal{G}'$ may be the variety of Abelian groups and $\mathcal{L}'$ may be the variety of distributive lattices.

Theorem 9.3. [19, Thm. 6.10] *If $\mathcal{V}$ is a variety and $\mathcal{W}$ is a polarized term idempotent variety, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. Let $t(x)$ be a polar term of $\mathcal{W}$. Define terms $p(x, y, z)$ and $q(x, y, z)$ as x and z respectively. Then the conditions of Theorem 8.6 take the following form:

(a) $\mathcal{V} \models x = x,\ y = y,$

(b) $\mathcal{W} \models t(x) = t(y).$

The identities of (a) are trivial, so (a) is satisfied. Since $t(x)$ is constant in $\mathcal{W}$, (b) is satisfied. □

Example 9.4. The variety $\mathcal{C}s$ of all constant semigroups is polarized and term idempotent. By Theorem 9.3, if $\mathcal{V}$ is a variety of magmas, then the Maltsev product $\mathcal{V} \circ \mathcal{C}s$ is a variety. E.g. $\mathcal{L}z \circ \mathcal{C}s$ is a variety.

The following result extends Theorem 1.5.

Theorem 9.5. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ be term idempotent. If the join $\mathcal{V} \vee \mathcal{W}$ is congruence 3–permutable, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. By Theorem 2.12, there are terms $p(x, y, z)$ and $q(x, y, z)$ such that $\mathcal{V} \vee \mathcal{W}$ satisfies the identities

$$x = p(x, y, y), \quad p(x, x, y) = q(x, y, y), \quad q(x, x, y) = y. \tag{9.1}$$

Since $\mathcal{V}$ and $\mathcal{W}$ also satisfy these identities, the conditions (a) and (b) of Theorem 8.6 are satisfied. □

Corollary 9.6. *If a variety $\mathcal{V}$ is term idempotent and congruence 3–permutable, then the Maltsev product $\mathcal{V} \circ \mathcal{V}$ is a variety.*

Corollary 9.7. *If $\mathcal{U}$ is an idempotent and congruence 3–permutable variety, then for every pair of subvarieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$, the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Maltsev was interested in families of subclasses of a class $\mathcal{K}$ of algebras, that are closed under the Maltsev $\mathcal{K}$–product. By Theorem 1.3, the class of subvarieties of a congruence permutable and polarized variety $\mathcal{U}$ is closed under the Maltsev $\mathcal{U}$–product. The following corollary provides another example of a variety $\mathcal{U}$ with such a property.

Corollary 9.8. *If $\mathcal{U}$ is an idempotent and congruence 3–permutable variety, then the class of subvarieties of $\mathcal{U}$ is closed under the Maltsev $\mathcal{U}$–product.*

Example 9.9. Let $\mathcal{P}_3$ denote the variety of the type $\{p, q\}$ with two ternary basic operation symbols that is defined by the identities (9.1). This is the most general congruence 3–permutable variety in the sense that a variety $\mathcal{V}$ is congruence 3–permutable iff there is an interpretation of $\mathcal{P}_3$ in $\mathcal{V}$. The variety $\mathcal{P}_3$ is also idempotent, so by Corollary 9.7, the Maltsev product of any pair of subvarieties of $\mathcal{P}_3$ is a variety. In particular $\mathcal{P}_3 \circ \mathcal{P}_3$ is a variety.

The special case of Theorem 9.1 when $\mathcal{W}$ is an idempotent variety already follows from the previous version of the sufficient condition presented in [19, Thm. 4.1]. However neither the full version of Theorem 9.1 nor Theorems 9.3 and 9.5 follow from [19, Thm. 4.1].

We will now investigate the consequences of Theorem 8.6 for varieties $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{V} \wedge \mathcal{W}$ is trivial. We previously observed that there is the following special case of the conditions (a) and (b).

Theorem 9.10. [19, Cor. 5.2] *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ be term idempotent. If there exist terms $p(x, y)$ and $q(x, y)$ such that*

(a) $\mathcal{V} \models p(x, y) = x,\ q(x, y) = y$,

(b) $\mathcal{W} \models p(x, y) = q(x, y)$,

then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

The assumptions of Theorem 9.10 may be rephrased as requiring the existence of $\mathcal{W}$–equivalent terms $p(x, y)$ and $q(x, y)$ that are $\mathcal{V}$–equivalent to x and y respectively. The existence of such terms implies the equivalence of the terms x and y in the meet $\mathcal{V} \wedge \mathcal{W}$. This in turn implies that $\mathcal{V} \wedge \mathcal{W}$ is trivial. If $\mathcal{V}$ and $\mathcal{W}$ are varieties that satisfy these assumptions, then

Theorem 7.14 provides a description of the $\mathcal{W}$–replica congruence of any algebra in the variety $\mathcal{V} \circ \mathcal{W}$. The following proposition shows that for a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, Theorem 9.10 presents the most general sufficient condition for the Maltsev product $\mathcal{V} \circ \mathcal{W}$ to be a variety which is provable using the method provided by Proposition 8.1.

Proposition 9.11. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, and let $\mathcal{W}$ be term idempotent and distinct from $\mathcal{A}$. The following conditions are equivalent.*

(i) *$P(n)$ holds for every $n \geq 1$.*

(ii) *$P(2)$ holds.*

(iii) *$\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ has simple $\mathcal{W}$–replica congruences.*

(iv) *There exist terms $p(x, y)$ and $q(x, y)$ such that*

  (a) *$\mathcal{V} \models p(x, y) = x,\ q(x, y) = y$,*

  (b) *$\mathcal{W} \models p(x, y) = q(x, y)$.*

Proof. Clearly (i) implies (ii). By Proposition 8.4, (ii) and (iii) are equivalent, and by Proposition 7.13, (iii) and (iv) are equivalent. The conditions (a) and (b) of (iv) are a special case of the conditions (a) and (b) of Theorem 8.6, so (iv) implies (i). □

Example 9.12. A group G is *Boolean* if every element of G is its own inverse. The variety $\mathcal{B}g$ of all Boolean groups can be considered as the subvariety of $\mathcal{S}g$ defined relative to $\mathcal{S}g$ by the identities

$$x \cdot (y \cdot y) = x = (y \cdot y) \cdot x.$$

Let $p(x, y)$ be the term $x \cdot (y \cdot y)$ and $q(x, y)$ be the term $(x \cdot x) \cdot y$. In the variety $\mathcal{B}g$, p and q are equivalent to x and y respectively. In the term idempotent variety $\mathcal{R}s$ of Example 6.3, p and q are equivalent. Thus, by Theorem 9.10, $\mathcal{B}g \circ \mathcal{R}s$ is a variety. The variety $\mathcal{B}g$ is strongly irregular and the variety $\mathcal{R}s$ is irregular, but not strongly irregular, so by Corollary 4.14, the variety $\mathcal{B}g \circ \mathcal{R}s$ is irregular.

Example 9.13. Consider the variety $\overline{\mathcal{R}s}$ of the type $\{\cdot, +\}$ defined by the identities defining

$\mathcal{R}s$ and the identity $x + y = x \cdot y$. This variety is equivalent to $\mathcal{R}s$. Let $p(x, y)$ be the term $x + (x \cdot y)$ and $q(x, y)$ be the term $(x \cdot y) + y$. These terms are $\overline{\mathcal{R}s}$–equivalent. In the variety $\mathcal{L}$ of lattices, p and q are equivalent to x and y respectively (due to the absorption laws). Thus, by Theorem 9.10, $\mathcal{L} \circ \overline{\mathcal{R}s}$ is a variety.

Varieties $\mathcal{V}$ and $\mathcal{W}$ are *independent* if there exists a term $p(x, y)$ that is $\mathcal{V}$–equivalent to x and $\mathcal{W}$–equivalent to y. Such a term is called a *decomposition term*. For independent varieties $\mathcal{V}$ and $\mathcal{W}$, the meet $\mathcal{V} \wedge \mathcal{W}$ is the trivial variety and the join $\mathcal{V} \vee \mathcal{W}$ consists of all algebras that are isomorphic to products $A \times B$ for $A \in \mathcal{V}$ and $B \in \mathcal{W}$. See [21, Sec. 3.5] for the discussion of independence of varieties.

Corollary 9.14. [19, Cor. 5.5] *If varieties $\mathcal{V}$ and $\mathcal{W}$ are independent and $\mathcal{W}$ is term idempotent, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. Let $p(x, y)$ be a decomposition term for $\mathcal{V}$ and $\mathcal{W}$. Define a term $q(x, y)$ as y. Then the conditions of Theorem 9.10 take the following form:

(a) $\mathcal{V} \models p(x, y) = x,\ y = y$,

(b) $\mathcal{W} \models p(x, y) = y$.

Since $p(x, y)$ is a decomposition term for $\mathcal{V}$ and $\mathcal{W}$, (a) and (b) are satisfied. □

Example 9.15. [19, Ex. 5.6] The varieties $\mathcal{L}z$ and $\mathcal{R}z$ are independent with a decomposition term $x \cdot y$, so by Corollary 9.14, $\mathcal{L}z \circ \mathcal{R}z$ is a variety. The join $\mathcal{L}z \vee \mathcal{R}z$ is the variety $\mathcal{R}b$ (see [12, p. 120]).

We will say that a term $p(x, y)$ is *symmetric* in a variety $\mathcal{V}$ if $\mathcal{V} \models p(x, y) = p(y, x)$.

Corollary 9.16. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, and let $\mathcal{W}$ be term idempotent. If there exists a term $p(x, y)$ that is symmetric in $\mathcal{W}$ and equivalent to x in $\mathcal{V}$, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. Let $p(x, y)$ be a term as in the statement of this corollary. Define a term $q(x, y)$ as $p(y, x)$. For such terms p and q, the conditions (a) and (b) of Theorem 9.10 are satisfied. □

Example 9.17. Recall from Example 6.16 that $\mathcal{C}om$ is the variety of commutative magmas and that $\mathcal{C}om^{\triangledown}$ is defined relative to $\mathcal{C}om$ by the identity $(x \cdot y) \cdot (x \cdot y) = x \cdot y$. Since the term $x \cdot y$ is symmetric in $\mathcal{C}om$ and it is equivalent to x in $\mathcal{L}z$, by Corollary 9.16, $\mathcal{L}z \circ \mathcal{C}om^{\triangledown}$ is a variety. By Corollary 6.19, this variety is term idempotent.

Example 9.18. Let $\mathcal{CI}$ be the variety of commutative and idempotent magmas. It is defined by the identities $x \cdot y = y \cdot x$ and $x \cdot x = x$. Let us consider an equivalent variety $\overline{\mathcal{CI}}$ of the type $\{\,\cdot\,, ^{-1}\,\}$ defined by the identities that define $\mathcal{CI}$ and the identity $x^{-1} = x$. The term $x \cdot (y \cdot y^{-1})$, which is equivalent to x in the variety $\mathcal{G}$ of groups of the type $\{\,\cdot\,, ^{-1}\,\}$ (see Example 9.2), is symmetric in $\overline{\mathcal{CI}}$, because

$$\overline{\mathcal{CI}} \models x \cdot (y \cdot y^{-1}) = x \cdot (y \cdot y) = x \cdot y = y \cdot x = y \cdot (x \cdot x) = y \cdot (x \cdot x^{-1}).$$

Thus, by Corollary 9.16, $\mathcal{G} \circ \overline{\mathcal{CI}}$ is a variety. The same is true for any subvariety of $\mathcal{CI}$. The variety $\mathcal{S}q$ of *Steiner quasigroups* is defined relative to $\mathcal{CI}$ by the identity $x \cdot (x \cdot y) = y$. Hence $\mathcal{G} \circ \overline{\mathcal{S}q}$ is a variety.

Theorem 9.19. [3, Thm. 6.3] *If $\mathcal{V}$ is a strongly irregular variety, then the Maltsev product $\mathcal{V} \circ \mathcal{S}$ is a variety.*

Proof. Let $t(x, y) = x$ be a strongly irregular identity true in $\mathcal{V}$. Since the term $t(x, y)$ contains both variables x and y, the identity $t(x, y) = t(y, x)$ is regular, and thus it is true in $\mathcal{S}$. By Corollary 9.16, $\mathcal{V} \circ \mathcal{S}$ is a variety. $\square$

Let $\mathcal{V}$ be a strongly irregular variety. Since $\mathcal{S}$ is regular, by Proposition 4.13, the variety $\mathcal{V} \circ \mathcal{S}$ is regular. Example 4.5 provides an equational base for $\mathcal{V} \circ \mathcal{S}$, which coincides with the equational base provided in [3]. As shown in Example 7.15, for any $A \in \mathcal{V} \circ \mathcal{S}$, the $\mathcal{S}$–replica congruence of A is given by

$$\varrho_A^{\mathcal{S}} \;=\; \{(a, b) \in A^2 \mid t(a, b) = a\},$$

where $t(x, y) = x$ is a strongly irregular identity satisfied in $\mathcal{V}$. If the type of $\mathcal{V}$ is plural, then algebras in $\mathcal{V} \circ \mathcal{S}$ are called *semilattice sums of $\mathcal{V}$–algebras* (see [20]).

The following counterexample shows that in Theorem 9.19, the assumption that $\mathcal{V}$ is strongly irregular cannot be replaced by the assumption that $\mathcal{V}$ is irregular. This answers in the negative the question posed in [3, Prob. 6.5].

Counterexample 9.20. [19, Ex. 5.9] Recall that the variety $\mathcal{C}s$ of constant semigroups is irregular, but not strongly irregular. We will show that $\mathcal{C}s \circ \mathcal{S}$ fails to be a variety by providing an example of an algebra $A \in \mathcal{C}s \circ \mathcal{S}$ that has a quotient $A/\theta \notin \mathcal{C}s \circ \mathcal{S}$. Let A be defined by the following table

$\cdot$	a	e	b	f
a	e	e	b	f
e	e	e	f	f
b	b	f	f	f
f	f	f	f	f

The $\mathcal{S}$–replica congruence of A has two congruence classes $\{a, e\}$ and $\{b, f\}$ that are constant semigroups with constant values e and f respectively, so $A \in \mathcal{C}s \circ \mathcal{S}$. The congruence θ of A with congruence classes $\{a\}$, $\{b\}$, and $E = \{e, f\}$ yields the quotient A/θ with the following table

$\cdot$	$\{a\}$	$\{b\}$	E
$\{a\}$	E	$\{b\}$	E
$\{b\}$	$\{b\}$	E	E
E	E	E	E

The $\mathcal{S}$–replica congruence of A/θ is the maximum congruence, i.e. the whole universe A/θ forms a congruence class. However A/θ is not a constant semigroup, so $A/\theta \notin \mathcal{C}s \circ \mathcal{S}$.

One might conjecture that for a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, the sufficient condition for $\mathcal{V} \circ \mathcal{W}$ to be a variety provided by Theorem 9.10 is also a necessary condition. If this was true, then the proposition that $\mathcal{V} \circ \mathcal{S}$ is a variety if $\mathcal{V}$ is irregular, which we refuted by a counterexample, could be refuted on account of the following result.

Proposition 9.21. *Let $\mathcal{V}$ and $\mathcal{W}$ be nontrivial varieties of a type that contains basic operations of arity at least two. If $\mathcal{V}$ is not strongly irregular and $\mathcal{W}$ is regular, then there do not exist terms $p(x, y)$ and $q(x, y)$ such that*

(a) $\mathcal{V} \models p(x, y) = x,\ q(x, y) = y$,

(b) $\mathcal{W} \models p(x, y) = q(x, y)$.

Proof. Assume that there exist terms p and q such that (a) and (b) hold. Then $\mathrm{var}(p) = \{x\}$, because otherwise it would be possible to derive a strongly irregular identity from $\mathcal{V} \models p = x$. Analogously, $\mathrm{var}(q) = \{y\}$. Hence $\mathcal{W} \not\models p = q$, which contradicts (b). $\square$

If the aforementioned conjecture is true, then for any nontrivial varieties $\mathcal{V}$ and $\mathcal{W}$ of a type that contains basic operations of arity at least two, such that $\mathcal{V}$ is not strongly irregular, $\mathcal{W}$ is regular and term idempotent, and $\mathcal{V} \wedge \mathcal{W}$ is trivial, the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is not a variety. An example of such varieties whose Maltsev product is nonetheless a variety would thus be a counterexample to the conjecture and the nonexistence of such examples would point towards its verity.

For any plural type Ω without symbols of basic operations of arity greater than two, let us define a variety $\mathcal{B}_\Omega$ by the following identities

(1) the associative and the idempotent law for a chosen binary $\cdot \in \Omega$,

(2) $x \bullet y = x \star y$ for all binary $\bullet, \star \in \Omega$,

(3) $f(x) = x$ for all unary $f \in \Omega$.

Then (1) makes $\cdot$ a band operation, (2) makes all binary basic operations equal to $\cdot$, so in particular it makes no difference which basic operation was chosen in (1), and (3) makes all unary basic operations equal to the identity operation. The variety $\mathcal{B}_\Omega$ is equivalent to the variety of bands $\mathcal{B}$. Algebras in $\mathcal{B}_\Omega$ may be called *Ω–bands*. We will denote $\mathcal{B}_\Omega$ simply by $\mathcal{B}$.

Corollary 9.22. [18, Thm. 4.4] *Let $\mathcal{V}$ be a variety of a plural type Ω without symbols of basic operations of arity greater than two. If there exist binary terms $p(x, y)$ and $q(x, y)$ such that the first variable of both terms is the same and the last variable of both terms is the same, and they are $\mathcal{V}$–equivalent to x and y respectively, then for any variety $\mathcal{W} \subseteq \mathcal{B}$, the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. The free band over generators x and y consists of the equivalence classes represented by terms x, y, xy, yx, xyx, and yxy. Each of the last 4 classes consists of all terms that contain both variables x and y, and have the same first variable and the same last variable as the representative term. Since the binary terms p and q contain both variables x and y and have the same first and the same last variable, they belong to the same class. Hence they are equivalent in $\mathcal{B}$. Therefore, by Theorem 9.10, $\mathcal{V} \circ \mathcal{W}$ is a variety. $\square$

Example 9.23. [18, Ex. 4.3] In the variety of groups $\mathcal{G}$, consider the binary terms $x \cdot (y^{-1} \cdot y)$ and $(x \cdot x^{-1}) \cdot y$. They have the same first variable and the same last variable, and $\mathcal{G}$ satisfies the identities

$$x \cdot (y^{-1} \cdot y) = x, \qquad (x \cdot x^{-1}) \cdot y = y. \tag{9.2}$$

By Corollary 9.22, $\mathcal{G} \circ \mathcal{B}$ is a variety. Furthermore, any variety that has the group basic operations satisfies the identities (9.2). E.g. for the variety $\mathcal{R}$ of all rings (defined without the symbols of constants), the Maltsev product $\mathcal{R} \circ \mathcal{B}$ is also a variety.

Example 9.24. [18, Ex. 4.1] In the variety of lattices $\mathcal{L}$, the terms $x + (x \cdot y)$ and $(x \cdot y) + y$ satisfy the conditions of Corollary 9.22, so $\mathcal{L} \circ \mathcal{B}$ is a variety. The same is true for any variety that has the lattice basic operations, e.g. the variety of Boolean algebras.

Example 9.25. [18, Ex. 4.2] In the variety of quasigroups $\mathcal{Q}$, the terms $(x \cdot y)/y$ and $x\backslash(x \cdot y)$ satisfy the conditions of Corollary 9.22, so $\mathcal{Q} \circ \mathcal{B}$ is a variety.

By Corollary 6.19, adding an assumption that $\mathcal{V}$ is idempotent to Theorem 8.6 or to one of its consequences presented in this chapter, results in a stronger conclusion that $\mathcal{V} \circ \mathcal{W}$ is a term idempotent variety. New examples of term idempotent varieties may be constructed this way. E.g. the following results are corollaries of Theorems 9.1 and 9.3 respectively.

Corollary 9.26. *If $\mathcal{V}$ is an idempotent congruence permutable variety and $\mathcal{W}$ is a term idempotent variety, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a term idempotent variety.*

Corollary 9.27. *If $\mathcal{V}$ is an idempotent variety and $\mathcal{W}$ is a polarized term idempotent variety, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a term idempotent variety.*

Example 9.28. The variety $\mathcal{S}$ of Ω–semilattices is idempotent and the variety $\mathcal{C}$ of constant algebras is polarized and term idempotent. Hence $\mathcal{S} \circ \mathcal{C}$ is a term idempotent variety.

Both Corollary 9.26 and Theorem 9.3 can be applied iteratively, which leads to the following results about repeated Maltsev products.

Corollary 9.29. *If $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are idempotent congruence permutable varieties and $\mathcal{W}$ is a term idempotent variety, then the repeated Maltsev product $\mathcal{V}_n \circ (\cdots \circ (\mathcal{V}_3 \circ (\mathcal{V}_2 \circ (\mathcal{V}_1 \circ \mathcal{W}))) \cdots)$ is a term idempotent variety.*

Corollary 9.30. *If $\mathcal{V}$ is a variety and $\mathcal{W}_1, \ldots, \mathcal{W}_n$ are polarized term idempotent varieties, then the repeated Maltsev product $(\cdots (((\mathcal{V} \circ \mathcal{W}_1) \circ \mathcal{W}_2) \circ \mathcal{W}_3) \circ \cdots) \circ \mathcal{W}_n$ is a variety.*

For a variety $\mathcal{V}$ and a positive integer n, let us define the nth *right–power* $\mathcal{V}^n$ of $\mathcal{V}$ and the nth *left–power* ${}^n\mathcal{V}$ of $\mathcal{V}$ as n–fold repeated Maltsev products of the following forms

$$\mathcal{V}^n = (\cdots (((\mathcal{V} \circ \mathcal{V}) \circ \mathcal{V}) \circ \mathcal{V}) \circ \cdots) \circ \mathcal{V},$$

$${}^n\mathcal{V} = \mathcal{V} \circ (\cdots \circ (\mathcal{V} \circ (\mathcal{V} \circ (\mathcal{V} \circ \mathcal{V}))) \cdots).$$

Corollaries 9.29 and 9.30 yield the following results.

Corollary 9.31. *If $\mathcal{V}$ is an idempotent congruence permutable variety, then every left–power of $\mathcal{V}$ is an idempotent variety.*

Corollary 9.32. *If $\mathcal{V}$ is a polarized term idempotent variety, then every right–power of $\mathcal{V}$ is a variety.*

We conclude this work with a brief discussion of possible directions of further research. The following are some of the open questions concerning our results.

Question 9.33. What is the most general sufficient condition for a Maltsev product of two varieties to be a variety which is provable using the method provided by Proposition 8.1?

Question 9.34. For a variety $\mathcal{V}$ and a term idempotent variety $\mathcal{W}$ such that $\mathcal{V} \wedge \mathcal{W}$ is trivial, is the sufficient condition for the Maltsev product $\mathcal{V} \circ \mathcal{W}$ to be a variety presented in Theorem 9.10 also a necessary condition?

If varieties $\mathcal{V}$ and $\mathcal{W}$ satisfy the sufficient condition of Theorem 8.6, then for any variety $\mathcal{U}$ that contains both $\mathcal{V}$ and $\mathcal{W}$, the Maltsev $\mathcal{U}$–product $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$ is a variety. This yields a sufficient condition for a Maltsev $\mathcal{U}$–product of two varieties to be a variety. However this sufficient condition lacks any requirements on the variety $\mathcal{U}$. It might prove to be fruitful to research the possibility of finding a common generalization of Theorem 8.6 and Theorem 1.4.

10 References

[1] C. Bergman, *Joins and Maltsev products of congruence permutable varieties*, Algebra Universalis **81**, Article no. 16, 2020.

[2] C. Bergman, *Universal Algebra: Fundamentals and Selected Topics*, CRC Press, Boca Raton, 2011.

[3] C. Bergman, T. Penza, A.B. Romanowska, *Semilattice sums of algebras and Mal'tsev products of varieties*, Algebra Universalis **81**, Article no. 33, 2020.

[4] W. J. Blok, P. Köhler, *The semigroup of varieties of generalized interior algebras*, Houston J. Math. Vol. **3**, No. 3, 1977.

[5] S. Burris, H.P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.

[6] B. Csákány, *Congruences and subalgebras*, Ann. Univ. Sei. Budapest Eötvös Sect. Math. **18**, 1975.

[7] R. Freese, R. McKenzie, *Maltsev families of varieties closed under join or Maltsev product*, Algebra Universalis **77**, 29–50, 2017.

[8] G. Grätzer, *Universal Algebra*, Springer-Verlag, New York, 1979.

[9] G. Grätzer, D. Kelly, *Products of lattice varieties*, Algebra Universalis **21**, 33-–45, 1985.

[10] G. Grätzer, D. Kelly, *The lattice variety $\mathcal{D} \circ \mathcal{D}$*, Acta Sci. Math. **51**, 73–80, 1987.

[11] J. Hagemann, A. Mitschke, *On n–permutable congruences*, Algebra Universalis **3**, Article no. 8, 8–12, 1973.

[12] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[13] A.A. Iskander, *Extension of algebraic systems*, Trans. Amer. Math. Soc. **281**, 309–327, 1984.

[14] P. Köhler, *The semigroup of varieties of Brouwerian semilattices*, Trans. Amer. Math. Soc. **241**, 331–342, 1978.

[15] A.I. Mal'tsev, *Multiplication of classes of algebraic systems*, Sibirsk. Mat. Zh. **8**, 346–365, 1967. (Russian language; English translation in: *The Metamathematics of Algebraic Systems. Collected Papers: 1936–1967*)

[16] R. McKenzie, G. McNulty, W. Taylor, *Algebras, Lattices, Varieties. Vol. 1*, Wadsworth & Brooks/Cole, Monterey, 1987.

[17] H. Neumann, *Varieties of Groups*, Springer Verlag, New York, 1967.

[18] T. Penza, A.B. Romanowska, *Mal'tsev products of varieties, I*, Algebra Universalis **82**, Article no. 33, 2021.

[19] T. Penza, A.B. Romanowska, *Mal'tsev products of varieties, II*, Algebra Universalis **83**, Article no. 21, 2022.

[20] J. Płonka, A.B. Romanowska, *Semilattice sums*, In: A.B. Romanowska, J.D.H. Smith (eds.) *Universal Algebra and Quasigroup Theory*, 123–158, Heldermann, Berlin, 1992.

[21] A.B. Romanowska, J.D.H. Smith, *Modes*, World Scientific, Singapore, 2002.